numbers

base-10 numbers

$$\begin{aligned} 12345 &= 1 \cdot 10^4 + 2 \cdot 10^3 + 3 \cdot 10^2 + 4 \cdot 10^1 + 5 \cdot 10^0 \\ 987.65 &= 9 \cdot 10^2 + 8 \cdot 10^1 + 7 \cdot 10^0 + 6 \cdot 10^{-1} + 5 \cdot 10^{-2} \end{aligned}$$

base-2 numbers

$$
\begin{aligned}
20_{\textsf{TEN}} \;\; (\text{or } 20_{10}) &= 11101_{\textsf{TWO}} \;\; (\text{or } 11101_2) \\
&= 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\
4_{\textsf{TEN}} &= 100_{\textsf{TWO}} \\
&= 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 \\
1.25_{\textsf{TEN}} &= 1.01_{\textsf{TWO}} \\
&= 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2}
\end{aligned}
$$

base-16 numbers

`0 1 2 3 4 5 6 7 8 9 A B C D E F`

$$15_{\text{TEN}} = \mathtt{F}_{\text{SIXTEEN}} = 15 \cdot 16^0$$

$$100_{\text{TEN}} = \mathtt{64}_{\text{SIXTEEN}} = 6 \cdot 16^1 + 4 \cdot 16^0$$

$$0.5_{\text{TEN}} = \mathtt{0.8}_{\text{SIXTEEN}} = 8 \cdot 16^{-1}$$

integers in C++

15_{TEN}	`15`
17_{EIGHT}	`017`
$\mathrm{F}_{\text{SIXTEEN}}$	`0xF`

99_{TEN}	`99`
143_{EIGHT}	`0143`
63_{SIXTEEN}	`0x63`

16_{TEN}	`16`
20_{EIGHT}	`020`
10_{SIXTEEN}	`0x10`

terminology

base-X number — X is the radix

I will call components of base X number 'digits'

- but not a great term — digit sometimes implies base-10
- sometimes "radit"
- base-2 digit = bit
- base-16 digit = nibble (sometimes)

base-10 = decimal

base-2 = binary

base-8 = octal

base-16 = hexadecimal

convert to decimal

$$42_{\mathsf{FIVE}} \quad =$$

$$121_{\mathsf{THREE}} \quad =$$

convert to decimal

$$
\begin{aligned}
42_{\text{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&=
\end{aligned}
$$

$$
121_{\text{THREE}} =
$$

convert to decimal

$$
\begin{aligned}
42_{\mathsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&= 20_{\mathsf{TEN}} + 2 = 22_{\mathsf{TEN}}
\end{aligned}
$$

$$
121_{\mathsf{THREE}} =
$$

convert to decimal

$$
\begin{aligned}
42_{\mathsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&= 20_{\mathsf{TEN}} + 2 = 22_{\mathsf{TEN}}
\end{aligned}
$$

$$
\begin{aligned}
121_{\mathsf{THREE}} &= 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 \\
&=
\end{aligned}
$$

convert to decimal

$$
\begin{aligned}
42_{\mathsf{FIVE}} &= 4 \cdot 5^1 + 2 \cdot 5^0 \\
&= 20_{\mathsf{TEN}} + 2 = 22_{\mathsf{TEN}}
\end{aligned}
$$

$$
\begin{aligned}
121_{\mathsf{THREE}} &= 1 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 \\
&= 9 + 6 + 1 = 16_{\mathsf{TEN}}
\end{aligned}
$$

convert to something (1)

$$42_{\mathsf{TEN}} \text{ as radix 5} \quad =$$

convert to something (1)

$$\begin{aligned}
42_{\text{TEN}} \text{ as radix } 5 &= \underline{\hspace{2em}}2 \\[1em]
42 \div 5 &= 8 + \ldots \\
42 \bmod 5 &= 2 \\
42 &= 8 \cdot 5 + 2
\end{aligned}$$

convert to something (1)

$$\begin{aligned} 42_{\text{TEN}} \text{ as radix 5} &= \_32 \\ \\ 42 \div 5 &= 8 + \ldots \\ 42 \bmod 5 &= 2 \\ 42 &= 8 \cdot 5 + 2 \\ 8 &= 1 \cdot 5 + 3 \end{aligned}$$

convert to something (1)

$$
\begin{aligned}
42_{\text{TEN}} \text{ as radix 5} &= 132_{\text{FIVE}} \\
\\
42 \div 5 &= 8 + \ldots \\
42 \bmod 5 &= 2 \\
42 &= 8 \cdot 5 + 2 \\
8 &= 1 \cdot 5 + 3 \\
1 &
\end{aligned}
$$

convert to something (2)

121_{TEN} as radix 11 $=$

convert to something (2)

$$121_{\text{TEN}} \text{ as radix } 11 = \underline{\hspace{2em}}0_{\text{ELEVEN}}$$

$$121 \div 11 = 11$$

$$121 \bmod 11 = 0$$

$$121 = 11 \cdot 11 + 0$$

convert to something (2)

$$
\begin{aligned}
121_{\mathsf{TEN}} \text{ as radix } 11 &= \_\,{\color{red}0}0_{\mathsf{ELEVEN}} \\
121 \div 11 &= 11 \\
121 \bmod 11 &= 0 \\
121 &= 11 \cdot 11 + 0 \\
11 &= 1 \cdot 11 + {\color{red}0}
\end{aligned}
$$

convert to something (2)

$$
\begin{aligned}
121_{\text{TEN}} \text{ as radix } 11 &= 100_{\text{ELEVEN}} \\
\\
121 \div 11 &= 11 \\
121 \text{ mod } 11 &= 0 \\
121 &= 11 \cdot 11 + 0 \\
11 &= 1 \cdot 11 + 0 \\
1 &
\end{aligned}
$$

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

$$
\begin{aligned}
uz_{\text{SIXTEEN}} &= u \cdot 16^1 + z \cdot 16^0 \\
&= (u_3 \cdot 2^3 + u_2 \cdot 2^2 + u_1 \cdot 2^1 + u_0 \cdot 2^0)2^4 + z_3 \cdot 2^3 + \ldots \\
&= u_3 \cdot 2^7 + u_2 \cdot 2^6 + u_1 \cdot 2^5 + u_0 \cdot 2^4 + z_3 \cdot 2^3 + \ldots \\
&= (u_3 u_2 u_1 u_0 z_3 z_2 z_1 z_0)_{\text{TWO}}
\end{aligned}
$$

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

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special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

1101	1110	0011	0000_{TWO}

special case: base-16 to base-2

each "nibble" (hexadecimal digit) = 4 binary bits

1	2	3	4_{SIXTEEN}
0001	0010	0011	0100_{TWO}

1101	1110	0011	0000_{TWO}
C	D	3	0_{SIXTEEN}

a note on bytes

one byte = one "octet" =
two nibbles (hexadecimal digits) =
eight bits

this class — byte is always eight bits

(some very old machines called different sizes "bytes")

a note on bytes

one byte = one “octet” =
two nibbles (hexadecimal digits) =
eight bits

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(some very old machines called different sizes “bytes”)

a note on bytes

one byte = one "octet" =
two nibbles (hexadecimal digits) =
eight bits

this class — byte is always eight bits

(some very old machines called different sizes "bytes")

exercise

$17_{\mathsf{NINE}} =?_{\mathsf{SEVEN}}$

exercise

$$17_{\mathsf{NINE}} = ?_{\mathsf{SEVEN}}$$

$$17_{\mathsf{NINE}} = 7 + 9 = 2 \cdot 7 + 2$$

$$17_{\mathsf{NINE}} = 22_{\mathsf{SEVEN}}$$

on math in other bases

you can do math in other bases

usually makes most sense for base 2...

		1	1	1	
	1	2	3	4	4_{SIXTEEN}
	×			1	5_{SIXTEEN}
	5	B	0	5	4
1	2	3	4	4	
1	7	E	4	9	4_{SIXTEEN}

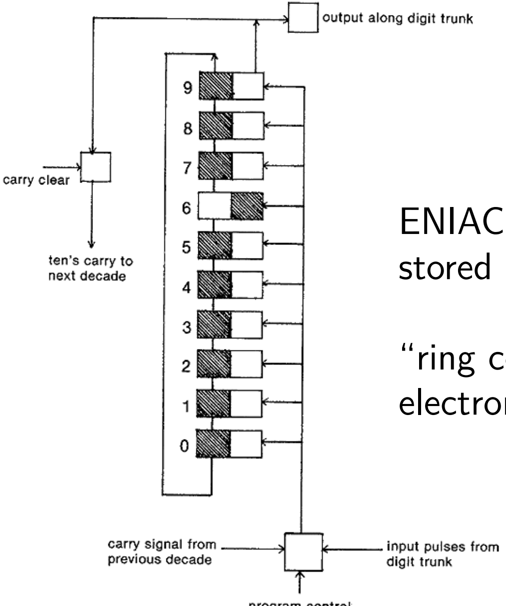
```
$ python3 -c 'print("{:x}".format(0x12344*0x15))'
17e494
```

integer representation

modern machine represent integers as series of bits (base-2)

why not base-10?

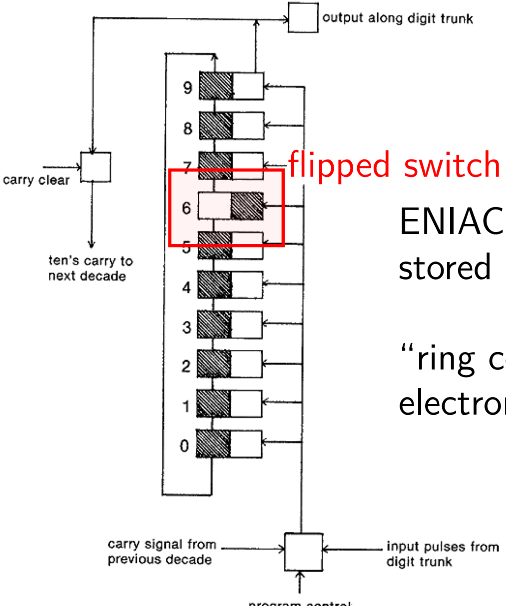
ENIAC: base-10 representation



ENIAC: 1946 computer
stored base-10 digits

"ring counter" of ten
electronic switches per digit

ENIAC: base-10 representation



ENIAC: 1946 computer
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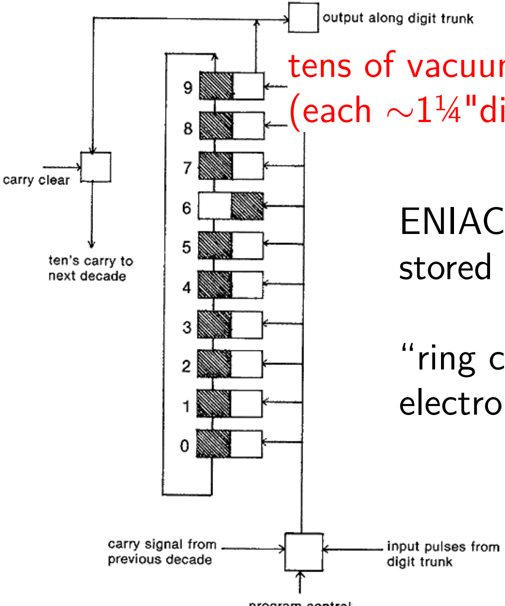
ENIAC: base-10 representation



tens of vacuum tubes total
(each ~1¼"diameter by 2¾"height)

ENIAC: 1946 computer
stored base-10 digits

"ring counter" of ten
electronic switches per digit

base-2 representation

base 2 — each switch represents one "digit"

- much more efficient use of switches

used in some pre-ENIAC electronic computers

- Atanasoff-Berry computer (1937, Ohio State)
- Z3 (1941, German Laboratory for Aviation)

quote: "A Survey of Digital Computer Memory Systems", Proceedings of the Institute of Radio Engineers, October 1953

base-2 representation

base 2 — each switch represents one "digit"

- much more efficient use of switches

used in some pre-ENIAC electronic computers

- Atanasoff-Berry computer (1937, Ohio State)
- Z3 (1941, German Laboratory for Aviation)

why not used in ENIAC?

- Eckert (ENIAC designer), 1953: "Although [binary-based digit counters] were known at the time of the construction of the ENIAC, it was not used because it required stable resistors, which were then much more expensive than they are now."
- also, important to input/output decimal digits directly

quote: "A Survey of Digital Computer Memory Systems", Proceedings of the Institute of Radio Engineers, October 1953

base-2 bit addition

+	0	1
0	00	01
1	01	10

base-2 bit addition

+	0	1
0	00	01
1	01	10

exactly one set to 1 — result (w/o carry) is 1; otherwise 0

base-2 bit addition

+	0	1
0	00	01
1	01	10

exactly one set to 1 — result (w/o carry) is 1; otherwise 0

both set to 1 — carry is 1; otherwise 0

base-2 capacity

n-bit number: $b_{n-1}b_{n-2}b_{n-3}\ldots b_2 b_1 b_0$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

$$\leq \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^n - 1$$

base-2 capacity

$$
\begin{aligned}
n\text{-bit number:} \quad & b_{n-1}b_{n-2}b_{n-3}\ldots b_2 b_1 b_0 \\
= \quad & \sum_{i=0}^{n-1} b_i \cdot 2^i \\
\leq \quad & \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^n - 1
\end{aligned}
$$

base-2 capacity

$$
\begin{aligned}
n\text{-bit number:} \quad & b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0 \\
= \quad & \sum_{i=0}^{n-1} b_i \cdot 2^i \\
\leq \quad & \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^n - 1
\end{aligned}
$$

missing pieces:

- negative numbers?
- non-whole numbers?
- what is n?

base-2 capacity

n-bit number: $$b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0$$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

$$\leq \sum_{i=0}^{n-1} 1 \cdot 2^i = 2^n - 1$$

missing pieces:

- negative numbers?
- non-whole numbers?
- what is n?

integer size in C++

varies between machines

- compiler uses what makes most sense on each machine?

	size in bits	
type	minimum	on lab machines
`unsigned char`	8	8
`unsigned short`	16	16
`unsigned int`	16	32
`unsigned long`	32	64

integer size in C++

varies between machines

compiler uses what makes most sense on each machine?

type	size in bits: minimum	size in bits: on lab machines
`unsigned char`	8	8
`unsigned short`	16	16
`unsigned int`	16	32
`unsigned long`	32	64

"unsigned" — can't be negative (no ± sign)

integer size in C++

varies between machines

compiler uses what makes most sense on each machine?

type	size in bits: minimum	size in bits: on lab machines
`unsigned char`	8	8
`unsigned short`	16	16
`unsigned int`	16	32
`unsigned long`	32	64

minimum size required by standard for all C++ compilers
all allowed to be bigger

querying sizes in C++

```
#include <climits>  // C: <limits.h>
...
ULONG_MAX or UINT_MAX or USHRT_MAX or UCHAR_MAX
// e.g. USHRT_MAX == 65535 on lab machines
```

```
#include <limits>
...
std::numeric_limits<unsigned long>::max()
    // == ULONG_MAX
...
```

```
sizeof(unsigned long)  // number of *bytes*
    // == 8 on lab machines
...
```

numbering bits

option 1: n-bit number: $b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

option 2: n-bit number: $b_0b_1b_2\ldots b_{n-3}b_{n-2}b_{n-1}$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^{n-i-1}$$

numbering bits

option 1: n-bit number: $b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^i$$

option 2: n-bit number: $b_0b_1b_2\ldots b_{n-3}b_{n-2}b_{n-1}$

$$= \sum_{i=0}^{n-1} b_i \cdot 2^{n-i-1}$$

two viable ways to number bits

numbering bits

option 1: n-bit number: $$b_{n-1}b_{n-2}b_{n-3}\ldots b_2b_1b_0 = \sum_{i=0}^{n-1} b_i \cdot 2^i$$

option 2: n-bit number: $$b_0b_1b_2\ldots b_{n-3}b_{n-2}b_{n-1} = \sum_{i=0}^{n-1} b_i \cdot 2^{n-i-1}$$

two viable ways to number bits

does it matter which I use?

do I have a way to ask for bit i?

numbering bytes

option 1: 4-byte number: $B_3 B_2 B_1 B_0$

$$= \sum_{i=0}^{3} B_i \cdot 256^i$$

option 2: 4-byte number: $B_0 B_1 B_2 B_3$

$$= \sum_{i=0}^{3} b_i \cdot 256^{3-i}$$

numbering bytes

option 1: 4-byte number: $B_3 B_2 B_1 B_0$

$$= \sum_{i=0}^{3} B_i \cdot 256^i$$

option 2: 4-byte number: $B_0 B_1 B_2 B_3$

$$= \sum_{i=0}^{3} b_i \cdot 256^{3-i}$$

two viable ways to number bytes

numbering bytes

option 1: 4-byte number: $B_3B_2B_1B_0$

$$= \sum_{i=0}^{3} B_i \cdot 256^i$$

option 2: 4-byte number: $B_0B_1B_2B_3$

$$= \sum_{i=0}^{3} b_i \cdot 256^{3-i}$$

two viable ways to number bytes

does it matter which I use?

in memory, yes — each byte needs an address (number)

memory

memory (as 64-bit values)

addr.	value (64-bit)
0	123999
8	323232
16	434093
…	…

$$
\begin{aligned}
123999 = &\quad 1 \cdot 256^2 \\
+ &\quad 228 \cdot 256^1 \\
+ &\quad 95 \cdot 256^0
\end{aligned}
$$

memory

if big endian (as 8-bit values)

addr.	value (8-bit)
0	0
1	0
2	0
3	0
4	0
5	1
6	228
7	95
8	0
9	0
10	0
11	...
...	...

memory (as 64-bit values)

addr.	value (64-bit)
0	123999
8	323232
16	434093
...	...

if little endian (as 8-bit values)

addr.	value (8-bit)
0	95
1	228
2	1
3	0
4	0
5	0
6	0
7	0
8	160
9	238
10	4
11	...
...	...

$$
\begin{aligned}
123999 = \quad & 1 \cdot 256^2 \\
+ \quad & 228 \cdot 256^1 \\
+ \quad & 95 \cdot 256^0
\end{aligned}
$$

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

---

little endian (e.g. lab machine):

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

---

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

get pointer to byte with lowest address in `value`

little endian (e.g. lab m

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

unless you do something like this won't see endianness

little endian (e.g. la

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

use pointer to get ith byte of value
(cast to int to output as number, not character)

little endian:

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << " ";
    }
    ...
}
```

little endian (e.g.

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

little endian: byte 0 is least significant
(affects overall value the least)

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

big endian: byte 0 is most significant
(affects overall value the most)

---

little endian (e.g.

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

---

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

finding endianness in C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
int main() {
    unsigned long value = 0x0123456789ABCDEF;
    cout << hex << value << endl;
    unsigned char *ptr = (unsigned char*) &value;
    for (int i = 0; i < sizeof(unsigned long); ++i) {
        cout << (int) ptr[i] << "␣";
    }
    ...
}
```

but we don't write numbers in a different order based on which end we call "part 0"

little endian

```
123456789abcdef
ef cd ab 89 67 45 23 1
```

big endian:

```
123456789abcdef
1 23 45 67 89 ab cd ef
```

little versus big endian

little endian — least significant part has lowest address

i.e. index 0 is the one's place

big endian — most significant part has the lowest address

i.e. index $n-1$ is the one's place

endianness in the real world

today and this course: little endian is dominant

- e.g. x86, *typically* ARM

historically: big endian was dominant

- e.g. *typically* SPARC, POWER, Alpha, MIPS, …
- still commonly used for networking because of this

many architectures have switchable endianness

- e.g. ARM, SPARC, POWER, MIPS
- usually, OS chooses one endianness

middle endian

sometimes not just big/little endian

e.g. number bytes most to least significant as
5, 6, 7, 8, 1, 2, 3, 4

e.g. doubles on little-endian ARM

generally some sort of historical accident

- e.g. ARM floating point designed for big endian?

endianness is about addresses

endianness is about numbering,
not (necessairily) placement on the page

but, probably assume English order (left to right, etc.) if not otherwise specified

addr.	value
0	95
1	228
2	1
3	0
4	0
5	0
6	0
7	0
8	160
9	238
10	4
11	…

=

addr.	value
…	…
11	…
10	4
9	238
8	160
7	0
6	0
5	0
4	0
3	0
2	1
1	228

endianness and bit-order

we won't talk about bit order

because bits don't have addresses

if I say "bit 0", question: "numbering from least significant or most significant"?

nothing about how pointers, etc. work suggests either answer is correct

endianness and writing out bytes

`0x0102` in binary: `0000000100000010`

English's order — most significant first

bytes of `0x0102` in big endian:
(byte 0) `00000001` (byte 1) `00000010`

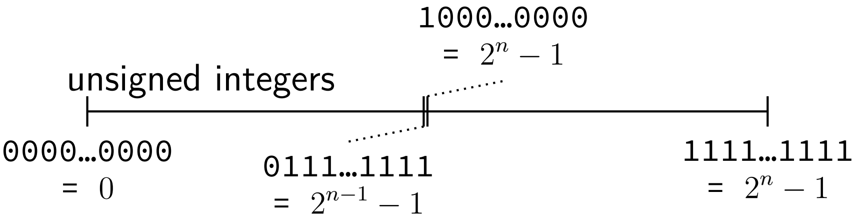
bytes of `0x0102` in little endian:
(byte 0) `00000010` (byte 1) `00000001`

usually, we don't change the order we write bits

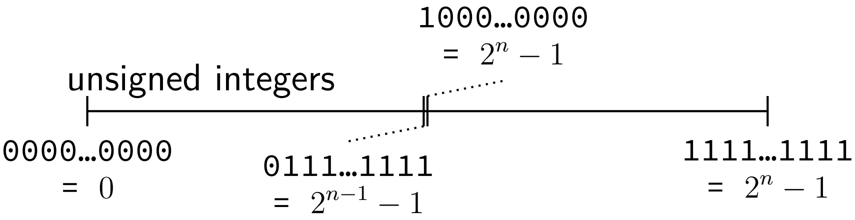
if writing out bytes, first in reading order is usually lowest address

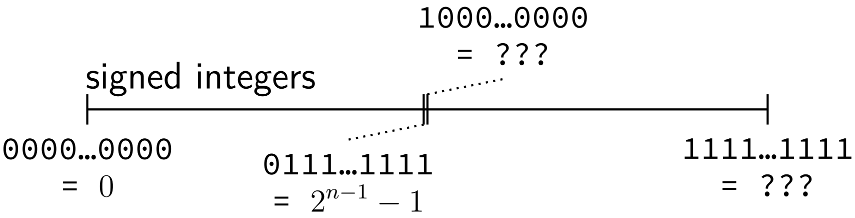
(we'll specify if not)

representing negative numbers

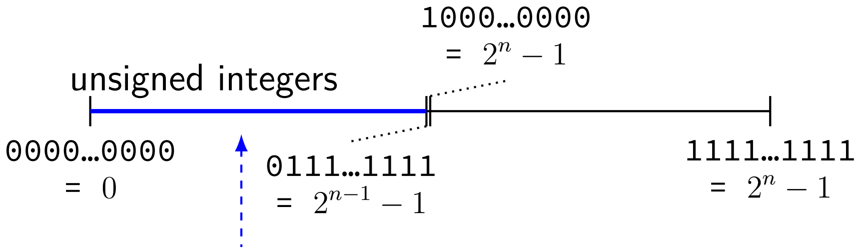


representing negative numbers



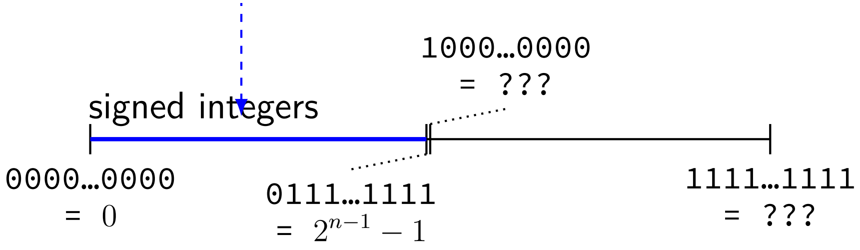


representing negative numbers

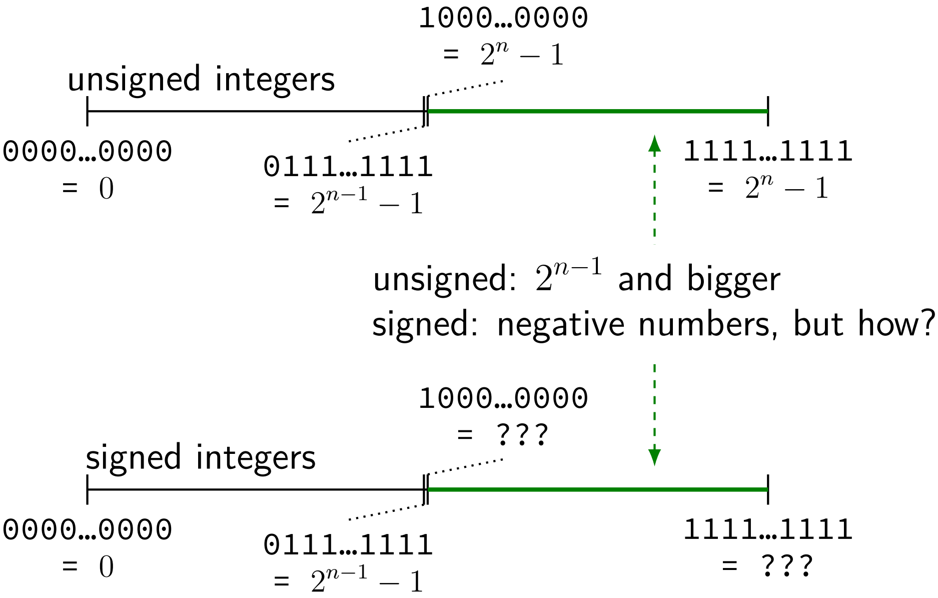


positive numbers up to $2^n - 1$

goal: same bits, signed or not

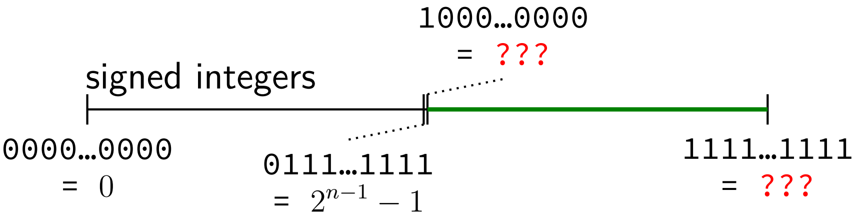


representing negative numbers



representing negative numbers

	sign & magnitude	1's complement	2's complement
`000…000`	0	0	`0`
`011…111`	$2^{n-1}-1$	$2^{n-1}-1$	$2^{n-1}-1$
`100…000`	0	$-2^{n-1}+1$	-2^{n-1}
`111…111`	$-2^{n-1}+1$	0	-1



representing negative numbers

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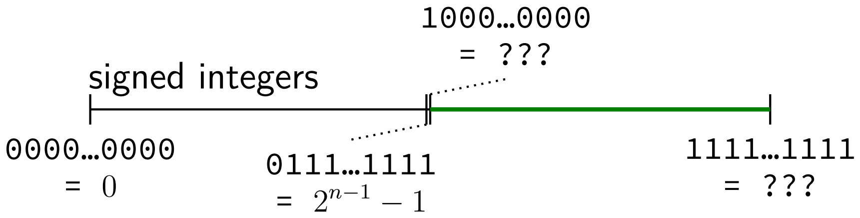
100…000	0	$-2^{n-1}+1$	-2^{n-1}
111…111	$-2^{n-1}+1$	0	-1

two representations of zero?

x == y needs to do something special



representing negative numbers

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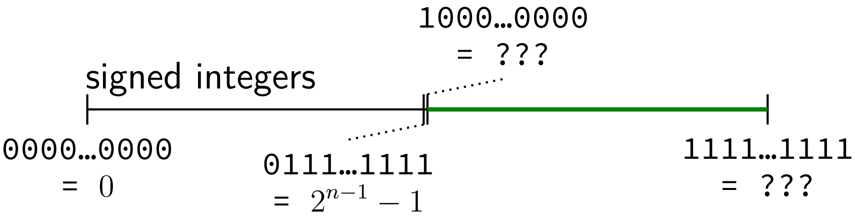
`100...000`	0	$-2^{n-1}+1$	-2^{n-1}
`111...111`	$-2^{n-1}+1$	0	-1

more negative values than positive values?



representing negative numbers

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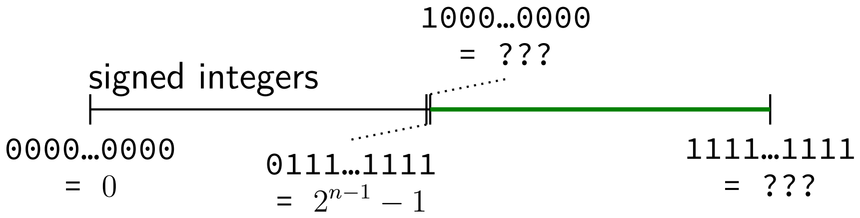
all 1's — least negative?



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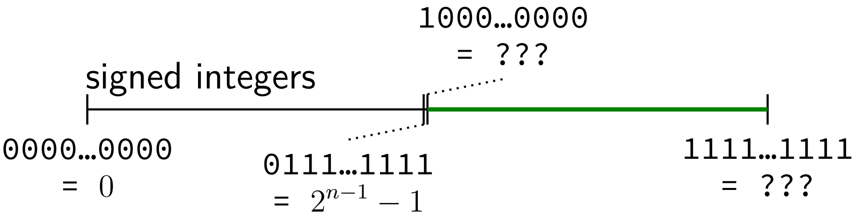
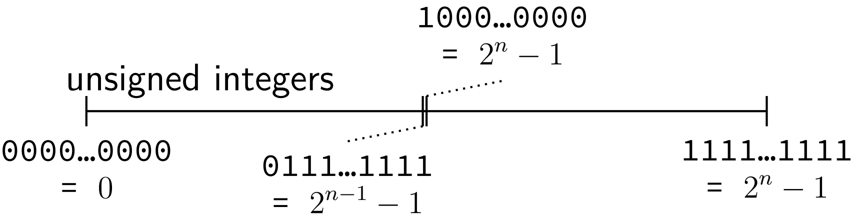
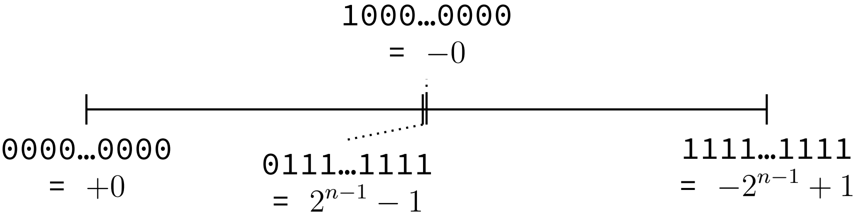
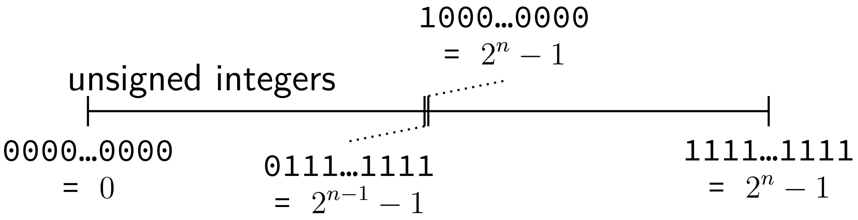


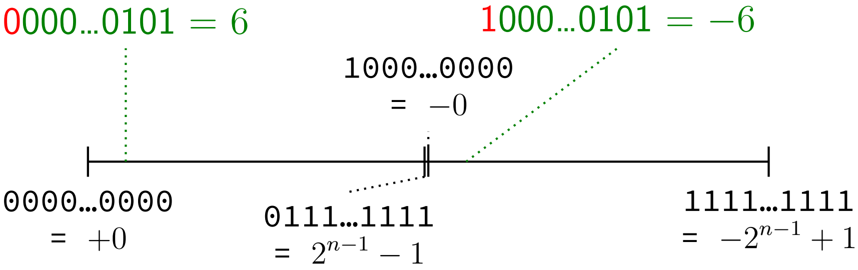
sign and magnitude





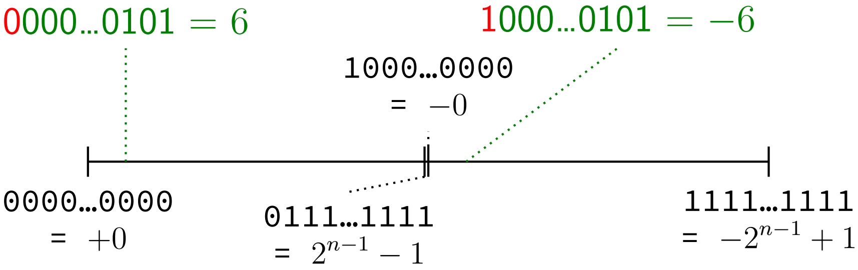
sign and magnitude





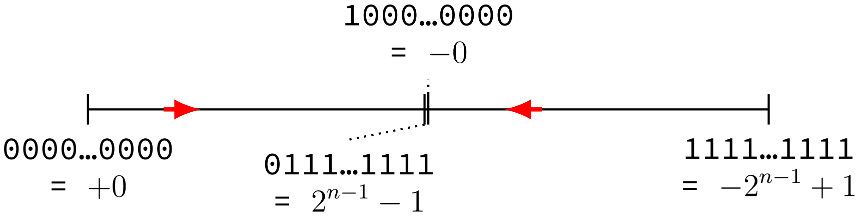
sign and magnitude

first bit is "sign bit" — 0 = positive, 1 = negative
flip sign bit to negate number

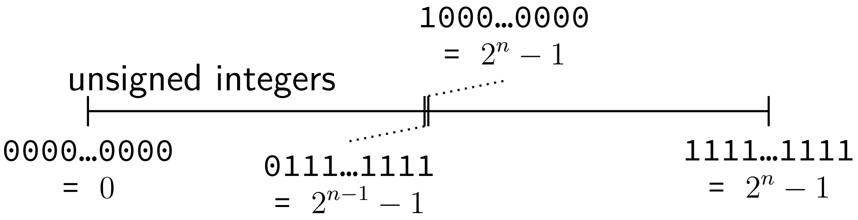


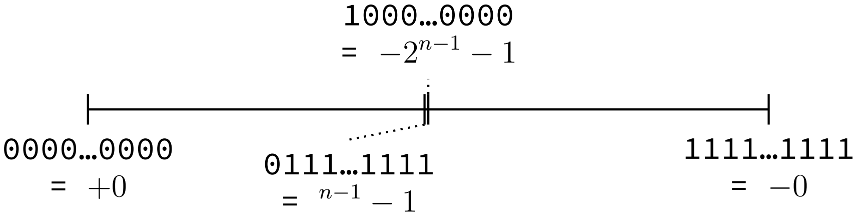
sign and magnitude

adding 1
different direction if negative

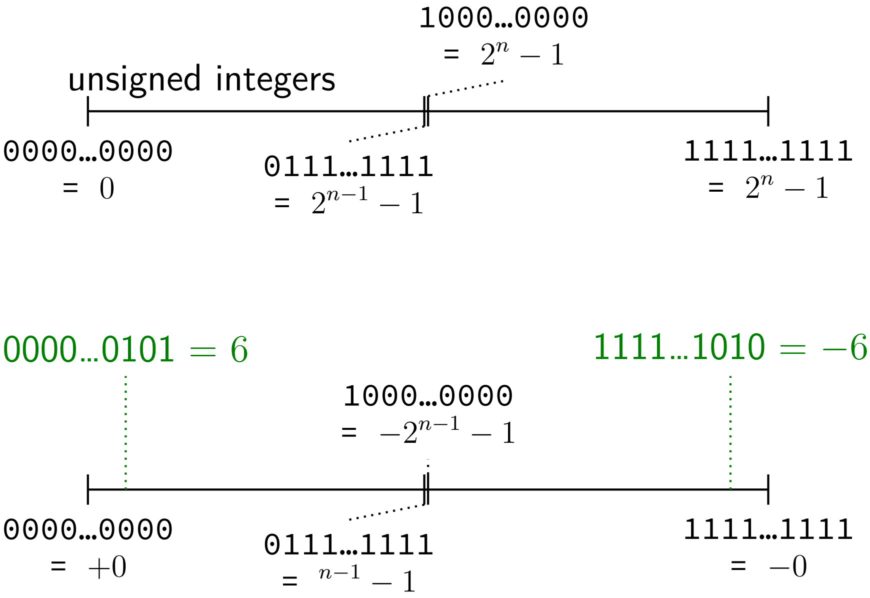


1's complement



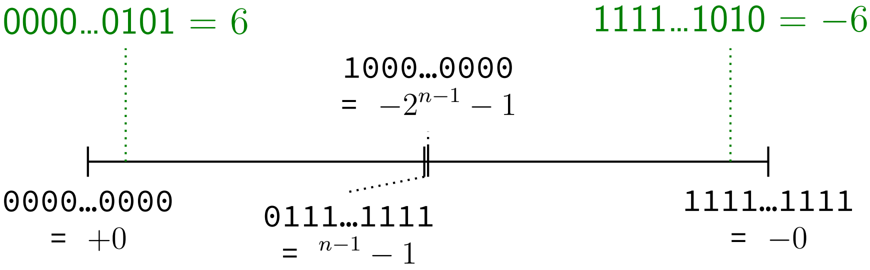


1's complement



1's complement

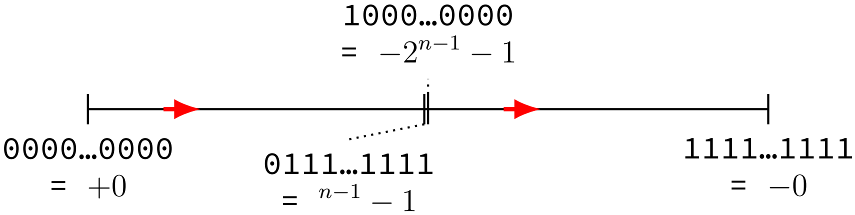
flip all bits to negate number



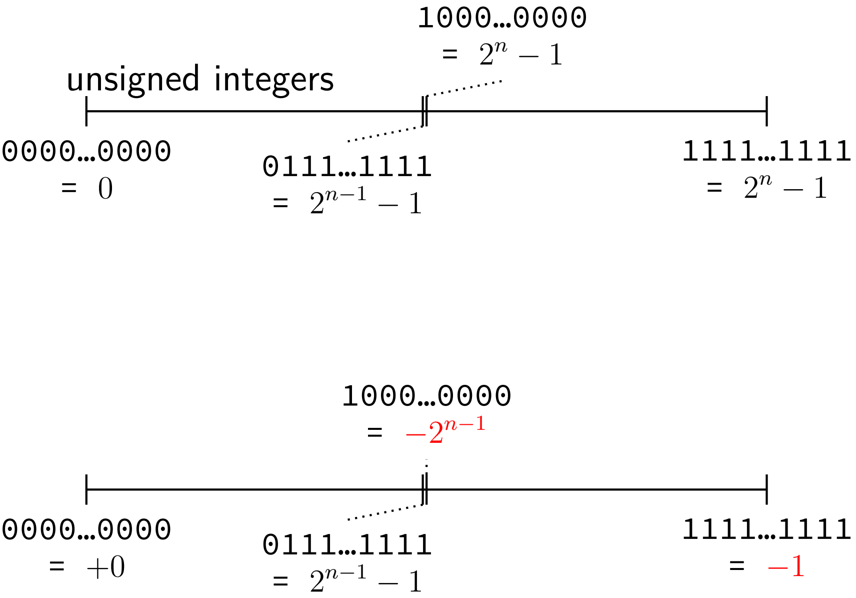
1's complement

adding 1

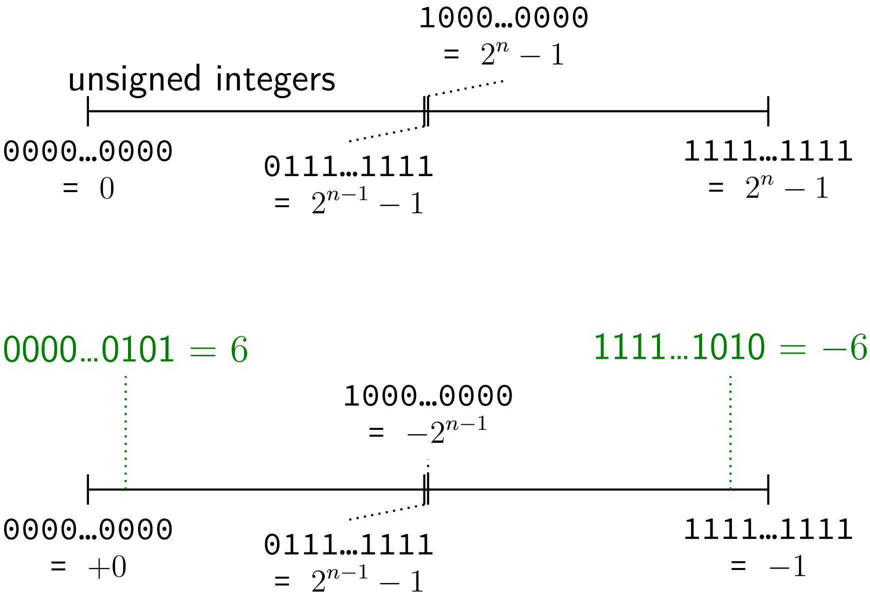
same direction, no matter original sign



two's complement

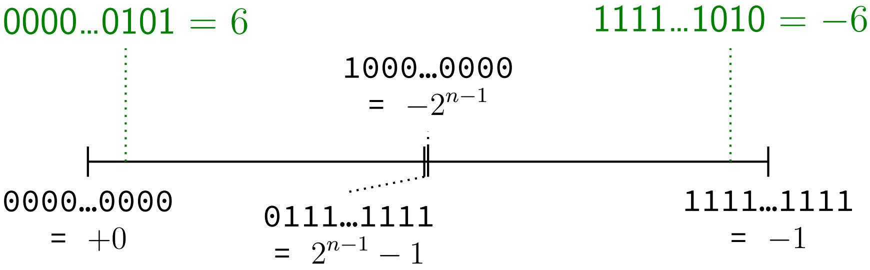


two's complement



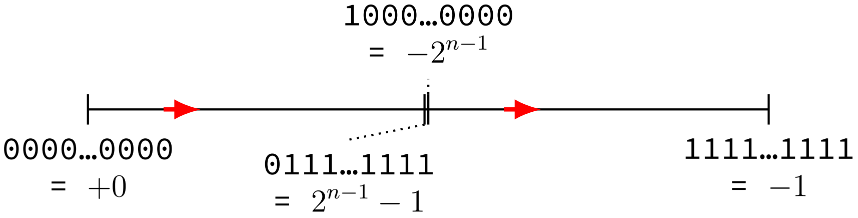
two's complement

flip all bits and add 1 to negate number



two's complement

adding 1
same direction, no matter original sign



2's complement (alt. perspective)

2's complement (5 bit)

$+10 =$	`0`		`1`		`0`		`1`		`0`	
	$0 \cdot (-2^4)$	+	$1 \cdot 2^3$	+	$0 \cdot 2^2 +$	+	$1 \cdot 2^1 +$	+	$0 \cdot 2^0$	
	0	+	2^3	+	0	+	2^1	+	$0 = 10$	
-10 $=$	`1`		`0`		`1`		`1`		`0`	
	$1 \cdot (-2^4)$	+	$0 \cdot 2^3$	+	$1 \cdot 2^2$	+	$1 \cdot 2^1$	+	$0 \cdot 2^0$	
	-2^4	+	0	+	2^2	+	2^1	+	$0 = -10$	

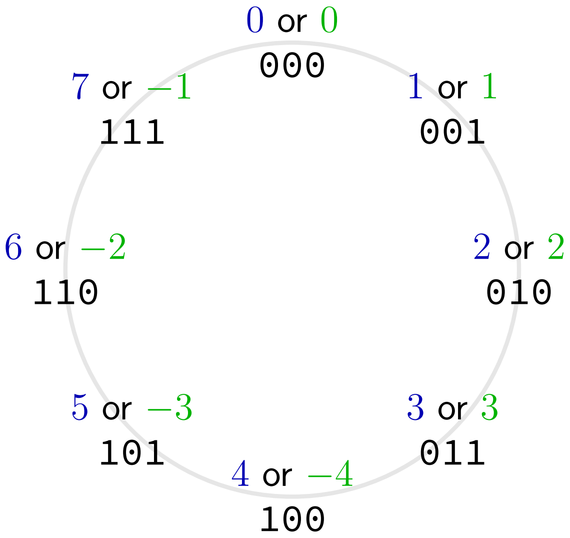
2's complement (alt. perspective)

2's complement (5 bit)

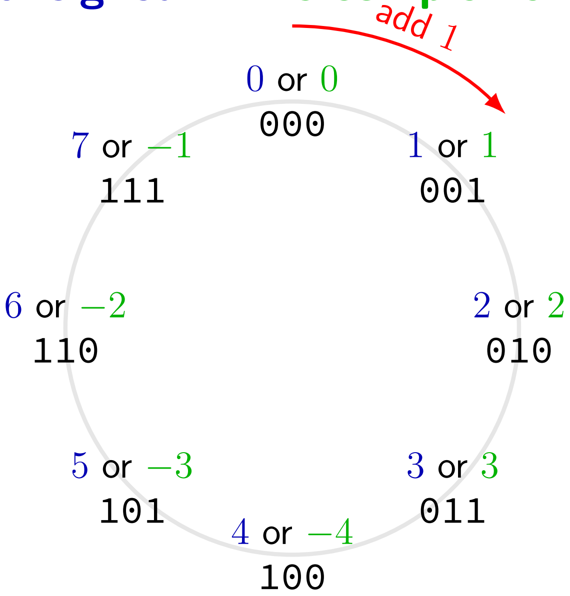
$+10 =$	`0`	`1`	`0`	`1`	`0`
	$0 \cdot (-2^4)$ +	$1 \cdot 2^3$ +	$0 \cdot 2^2 +$ +	$1 \cdot 2^1 +$ +	$0 \cdot 2^0$
	0 +	2^3 +	0 +	2^1 +	$0 = 10$
-10 =	`1`	`0`	`1`	`1`	`0`
	$1 \cdot (-2^4)$ +	$0 \cdot 2^3$ +	$1 \cdot 2^2$ +	$1 \cdot 2^1$ +	$0 \cdot 2^0$
	-2^4 +	0 +	2^2 +	2^1 +	$0 = -10$

"-2^4s place"

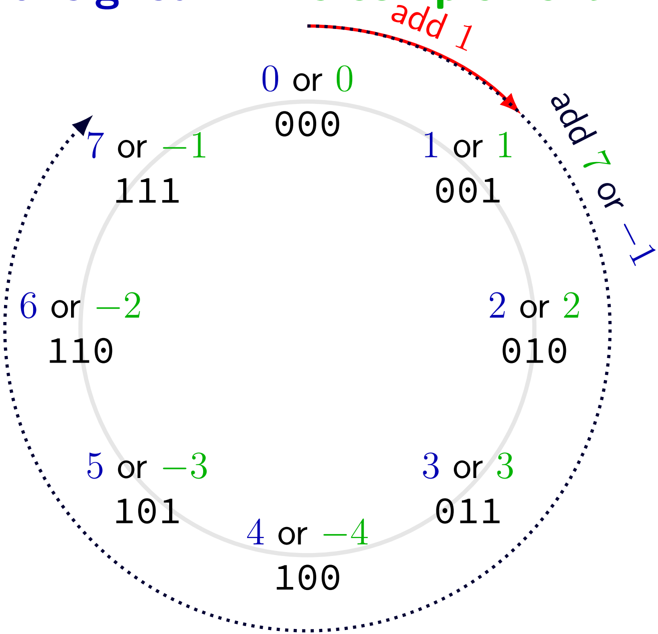
unsigned v. 2's complement



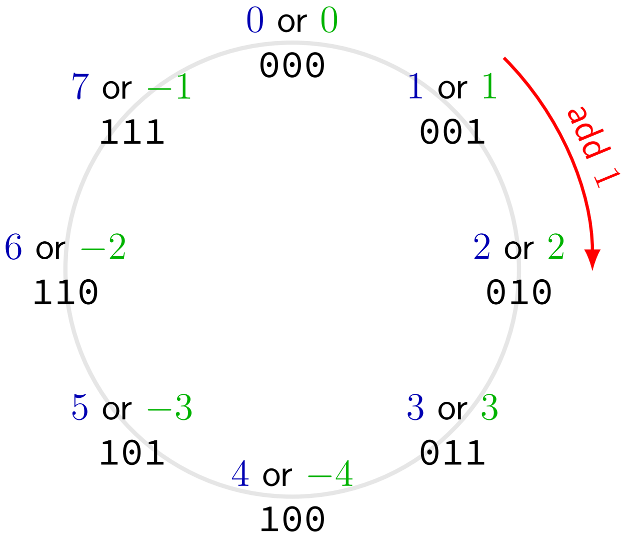
unsigned v. **2's complement**



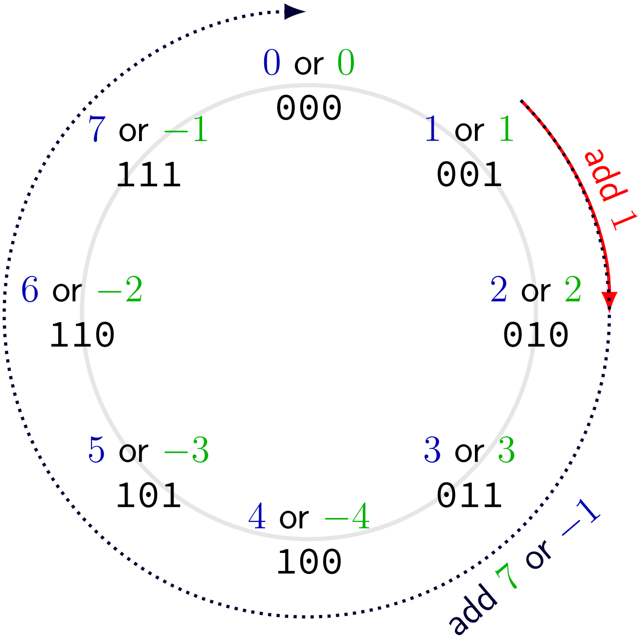
unsigned v. 2's complement



unsigned v. 2's complement



unsigned v. 2's complement



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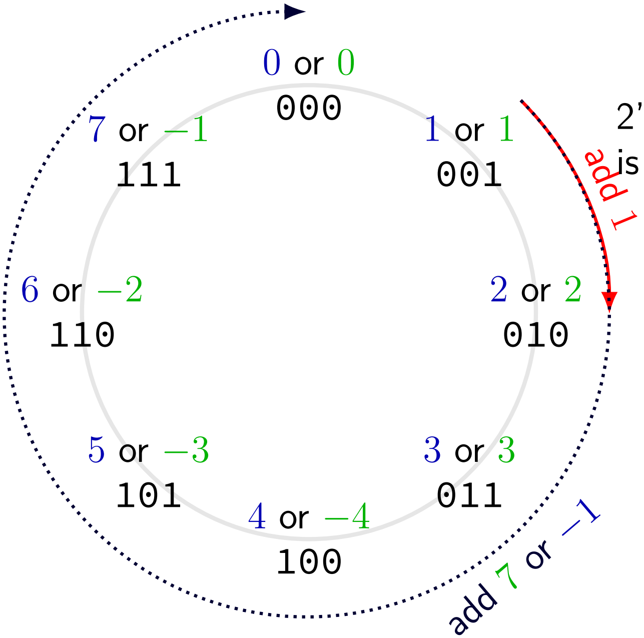


2's complement addition
is same as unsigned addition

other 2's complement arithmetic

subtraction also the same as unsigned

multiplication — repeated addition — mostly the same

(but need some extra precision)

converting to 2's complement (version 1)

take absolute value, convert to bits

if negative, flip all the bits and add one

$-14 \rightarrow -00001110 \rightarrow 11110001 + 1 \rightarrow 11110010$

$-127 \rightarrow -01111111 \rightarrow 10000000 + 1 \rightarrow 10000001$

$-128 \rightarrow -10000000 \rightarrow 01111111 + 1 \rightarrow 10000000$

converting to 2's complement (version 2)

if negative, take absolute value, subtract from 2^n, encode that

$-14 \to 2^8 - 14 = 242 \to 11110010$

$-127 \to 2^8 - 127 = 129 \to 10000001$

$-128 \to 2^8 - 127 = 129 \to 10000000$

sign extension

have 8-bit 2's complement number `1101 0111`

what is this as a 16-bit 2's complement number?

sign extension

have 8-bit 2's complement number `1101 0111`

what is this as a 16-bit 2's complement number?

general rule: extend by copying the *sign bit*:

```
1101 0111 → 1111 1111 1101 0111
0010 1111 → 0000 0000 0010 1111
```

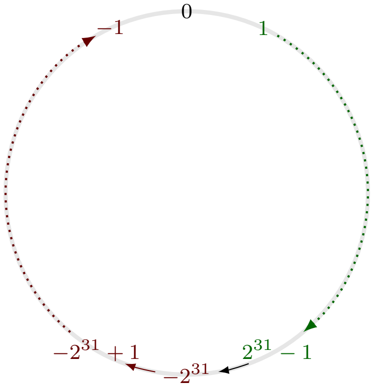
"sign extension"

two's complement summary

	-2^{31}	$+2^{30}$	$+2^{29}$		$+2^{2}$	$+2^{1}$	$+2^{0}$
$-1 =$	1	1	1	...	1	1	1

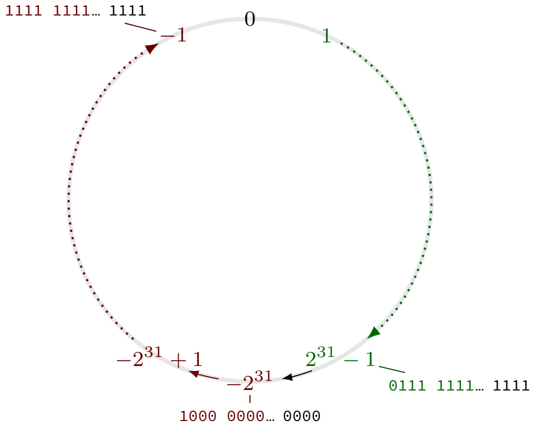
two's complement summary

	-2^{31}	$+2^{30}$	$+2^{29}$		$+2^{2}$	$+2^{1}$	$+2^{0}$
$-1 =$	1	1	1	...	1	1	1



two's complement summary

$$-1 = \overset{-2^{31}}{1}\quad \overset{+2^{30}}{1}\quad \overset{+2^{29}}{1}\quad \ldots\quad \overset{+2^{2}}{1}\quad \overset{+2^{1}}{1}\quad \overset{+2^{0}}{1}$$



integer overflow

"wrap around"

8-bit signed: $127 + 1 \rightarrow -128$

8-bit unsigned: $255 + 1 \rightarrow 0$

16-bit signed: $32\,767 + 1 \rightarrow -32\,768$

16-bit unsigned: $65\,536 + 1 \rightarrow 0$

32-bit signed: around 2 billion

64-bit signed: around 9×10^{18}

…

on integer overflow in C++ (1)

```
unsigned int x; // lab machines: 32-bit unsigned
x = 4294967295; // (2 to the 32) minus 1
x += 10;
cout << x << endl; // OUTPUT: 9
```

on integer overflow in C++ (1)

```
int x; // lab machines: 32-bit signed
x = 2147483647;  // mxaimum integer
x += 10;  // UNDEFINED!
cout << x << endl; // EXPECT big negative number,
                   // but not gaurenteed
```

in practice: usually get wraparound behavior…

but compiler is not required to do this for signed numbers

and takes advantage of this to optimize, sometimes

some real numbers

$$\frac{1}{3}$$

$$-\frac{100}{7}$$

$$\pi$$

$$0.1$$

$$\sqrt{2}$$

...

want to represent these: accurately? compactly? efficiently?

fixed point

$$\frac{1}{3} = 0.101010101\ldots_{\textsf{TWO}}$$

$$\approx +0000.1010_{\textsf{TWO}}$$ — represent as `00000 1010`

$$\frac{100}{7} = 1110.001001001\ldots_{\textsf{TWO}}$$

$$\approx -1110.0010_{\textsf{TWO}}$$ — represent as `01110 0010`

fixed point

$$\begin{aligned}\frac{1}{3} &= 0.101010101\ldots_{\text{TWO}} \\ &\approx +0000.1010_{\text{TWO}} \text{— represent as } \texttt{00000 1010} \\ \frac{100}{7} &= 1110.001001001\ldots_{\text{TWO}} \\ &\approx -1110.0010_{\text{TWO}} \text{— represent as } \texttt{01110 0010}\end{aligned}$$

$x \approx y/2^K$ — represent with fixed-sized signed integer y

this case: $y/2^4$ and y is 9 bits.

why fixed-point?

$x \approx y/2^K$ (y fixed-sized singed integer)

math similar to integer math:

- addition/subtraction — same
- multiplication — same except divide by 2^K
- division — same except multiply by 2^K

easy to understand what values are represented well

why not fixed-point?

pretty small range of numbers for space used

hard to choose a 2^K that works for lots of applications

recall (?): scientific notation

$$
\begin{aligned}
+\frac{1}{3} &= +0.33333333\ldots \\
&\approx +3.33 \cdot 10^{-1} \\
-\frac{100}{7} &= -14.285714\ldots \\
&\approx -1.42 \cdot 10^{+1}
\end{aligned}
$$

recall (?): scientific notation

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$\pm\text{mantissa} \cdot \text{base}^{\text{exponent}}$

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$\pm \text{mantissa} \cdot \text{base}^{\text{exponent}}$

base-2 scientific notation

$$
\begin{aligned}
\frac{1}{3} &= 0.101010101\ldots_{\textsf{TWO}} \\
&\approx 0.1010101010_{\textsf{TWO}} = +1.0101010101_{\textsf{TWO}} \cdot 2^{-1} \\
-\frac{125}{4} &= -111111.01\ldots_{\textsf{TWO}} \\
&= -1.1111101_{\textsf{TWO}} \cdot 2^{2} \\
-\frac{100}{7} &= -1110.01001001\ldots_{\textsf{TWO}} \\
&\approx -1110.010010_{\textsf{TWO}} = -1.1100100101_{\textsf{TWO}} \cdot 2^{3}
\end{aligned}
$$

IEEE half-precision floating point

$$-1.1100100101_{\text{TWO}} \cdot 2^{3}$$

IEEE half-precision floating point

$$-1.1100100101_{\text{TWO}} \cdot 2^{3}$$

sign (1 bit)

- 0 for $+$
- 1 for $-$

mantissa (10 bits)

don't store leading "1."
(because always present)

exponent (5 bits)

store $3 + 15 = 18$
15 is 'bias'

IEEE half-precision floating point

$-1.1100100101_{\text{TWO}} \cdot 2^3$

sign (1 bit)

0 for $+$
1 for $-$

mantissa (10 bits)

don't store leading "1."
(because always present)

exponent (5 bits)

store $3 + 15 = 18$
15 is 'bias'

```
1 10010 1100100101
```

IEEE half-precision floating point

$-1.1100100101_{\text{TWO}} \cdot 2^{3}$

sign (1 bit)

0 for $+$
1 for $-$

mantissa (10 bits)

don't store leading "1."
(because always present)

exponent (5 bits)

store $3 + 15 = 18$
15 is 'bias'

`1 10010 1100100101`

on typical little endian system:
byte 0: `00100101`
byte 1: `11001011`

IEEE half precision float

1 sign bit (1 for negative)

5 expontent bits

- bias of 15 — if bits as unsigned are e, exponent is $E = e - 15$

10 mantissa bits

- leading "1." not stored

$$\text{value} = (1 - 2 \cdot \text{sign}) \cdot (1.\text{mantissa}_{\text{TWO}}) \cdot 2^{\text{exponent}-15}$$

approximation

example: represented $\frac{100}{7} \approx 14.285$ as $\frac{1829}{128} \approx 14.289$

too large by $\frac{3}{896}$

10 bits mantissa + implicit "1" — about $\log_{10}(2^{11}) \approx 3.3$ decimal digits

other IEEE precisions

	half	single	double	quad
C++*/Java type	—	`float`	`double`	—
sign bits	1	1	1	1
exponent bits	5	8	11	15
exponent bias	15 $(2^5 - 1)$	127 $(2^7 - 1)$	1023 $(2^{10} - 1)$	16383 $(2^{14} - 1)$
mantissa bits	10	23	52	112
total bits	16	32	64	128

(* = typical C++ type; might vary in some implementations)

on exponent bias

general rule: $2^{\text{exponent bits}-1} - 1$

i.e. `0111...1` means 2^0

idea: best at representing numbers around 1

diversion: 25.25 to binary

$$
\begin{aligned}
25.25 &= 25 + \frac{1}{4} = \frac{101}{4} \\
&= \frac{1100101_{\mathsf{TWO}}}{2^2} \\
&= 11001.01_{\mathsf{TWO}}
\end{aligned}
$$

diversion: 25.25 to binary

$$
\begin{aligned}
25.25 &= 2^4 + (25.25 - 2^4) = 2^4 + 9.25 \\
&= 2^4 + 2^3 + (9.25 - 2^3) = 2^4 + 2^3 + 1.25 \\
&= 2^4 + 2^3 + (9.25 - 2^3) = 2^4 + 2^3 + 1.25 \\
& \quad (1.25 < 2^2) \\
& \quad (1.25 < 2^1) \\
&= 2^4 + 2^3 + (1.25 - 2^0) = 2^4 + 2^3 + 2^0 + 0.25 \\
& \quad (0.25 < 2^{-1}) \\
&= 2^4 + 2^3 + 2^0 + 2^{-2} + (0.25 - 2^{-2}) = 2^4 + 2^3 + 2^0 + 2^{-2}
\end{aligned}
$$

float example: manually (1)

$$25.25 = \frac{101}{4} = \frac{101}{2^2}$$

largest power of two < 25.25? $16 = 2^4$

(means $1 < 25.25/16 < 2$)

$$\begin{aligned}
\frac{101}{4} \cdot \frac{2^4}{2^4} &= \frac{101 \cdot 2^4}{2^6} \\
&= \frac{101}{2^6} \times 2^4 \\
&= \frac{1100101_{\text{TWO}}}{2^6} \times 2^4 \\
&= 1.100101_{\text{TWO}} \times 2^4
\end{aligned}$$

float example: manually (2)

$$25.25 = \frac{101}{4} = 11001.01_{\text{TWO}} =$$

$$+1.1001\,0100\,0000\,0000\,0000\,000_{\text{TWO}} \cdot 2^{4}$$

`float` example: manually (2)

$$25.25 = \frac{101}{4} = 11001.01_{\text{TWO}} =$$

$$+1.1001\,0100\,0000\,0000\,0000\,000_{\text{TWO}} \cdot 2^{4}$$

sign (1 bit)

0 for +

mantissa (23 bits)

(leading "1." not stored)

exponent (8 bits)

store "$4 + 127 = 1000\,0011_{\text{TWO}}$"

127 is bias for `float`

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127 is bias for `float`

```
0 1000 0011 1001 0100 0000 0000 0000 000
```

float example: from C++

```
#include <iostream>
using std::cout; using std::hex; using std::endl;
// union: all elements use the *same memory*
union floatOrInt {
    float f;
    unsigned int u;
};
int main() {
    union floatOrInt x;
    x.f = 25.25;
    cout << hex << x.u << endl;
// OUTPUT: 41ca0000
}
```

```
   4      1      c      a      0      0      0      0
 0100   0001   1100   1010   0000   0000   0000   0000
```

float example 2: manually

$$0.1_{\text{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\frac{1}{16} + \frac{1}{32} + \frac{1}{256} + 0.00234375 = \ldots$$
$$\ldots = 0.00011001100110011\ldots_{\text{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,101_{\text{TWO}} \cdot 2^{-4}$$

float example 2: manually

$$0.1_{\text{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$

$$\frac{1}{16} + \frac{1}{32} + \frac{1}{256} + 0.00234375 = \ldots$$

$$\ldots = 0.00011001100110011\ldots_{\text{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,101_{\text{TWO}} \cdot 2^{-4}$$

sign (1 bit)

0 for $+$

mantissa (23 bits)

last **1** from rounding

exponent (8 bits)

store "$-4 + 127 = 0111\,1011_{\text{TWO}}$"

float example 2: manually

$$0.1_{\textsf{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\frac{1}{16} + \frac{1}{32} + \frac{1}{256} + 0.00234375 = \ldots$$
$$\ldots = 0.00011001100110011\ldots_{\textsf{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,101_{\textsf{TWO}} \cdot 2^{-4}$$

sign (1 bit)

0 for +

mantissa (23 bits)

last 1 from rounding

exponent (8 bits)

store "$-4 + 127 = 0111\,1011_{\textsf{TWO}}$"

`0 0111 1011 1001 1001 1001 1001 1001 101`

float example 2: manually

$$0.1_{\text{TEN}} = \frac{1}{16} + 0.0375 = \frac{1}{16} + \frac{1}{32} + 0.00625 =$$
$$\frac{1}{16} + \frac{1}{32} + \frac{1}{256} + 0.00234375 = \ldots$$
$$\ldots = 0.0001100110011001\ldots_{\text{TWO}} \approx$$

$$+1.1001\,1001\,1001\,1001\,1001\,101_{\text{TWO}} \cdot 2^{-4}$$

sign (1 bit)

0 for +

mantissa (23 bits)

last 1 from rounding

exponent (8 bits)

store "$-4 + 127 = 0111\,1011_{\text{TWO}}$"

`0 0111 1011 1001 1001 1001 1001 1001 101`

closest `float` to 0.1 between 0.1 and 0.1000001

aside: binary long division

```
        0.000110011001100110011...
      ____________________________
1010 | 1.0000000000000000000000000
        1 010
          1100
          1010
            10000
             1010
               1100
                1010
                  10000
```

float example 2: inaccurate (1)

```
#include <iostream>
using std::cout; using std::endl;

int main(void) {
    int count;
    float base = 0.1f;
    for (count = 0; base * count < 10000000; ++count) {}
    cout << count << endl;
    // OUTPUT: 99999996
    return 0;
}
```

float example 2: inaccurate (2)

```
#include <iostream>
using std::cout; using std::endl;

int main(void) {
    int count = 0;
    for (float f = 0; f < 2000.0; f += 0.1) {
        ++count;
    }
    cout << count << endl;
    // OUTPUT: 20004
    return 0;
}
```

float example 2: inaccurate (3)

```
#include <iostream>
using std::cout; using std::endl;
int main(void) {
    cout.precision(30);
    for (float f = 0; f < 2000.0; f += 0.1) {
        cout << f << endl;
    }
    return 0;
}
```

```
0
0.100000001490116119384765625
0.200000002980232238769531250
…
2.2000000476837158203125
2.2999999523162841796875
…
```

float to number (1)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`1 1000 0000 1100 0000 0000 0000 0000 000` $=$???

float to number (1)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`1 1000 0000 1100 0000 0000 0000 0000 000` $= ???$

$-1.1100\ldots \cdot 2^{128-127=1} = -11.1 = -3.5_{\text{TEN}}$

float to number (1)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`1 1000 0000 1100 0000 0000 0000 0000 000` $= ???$

$-1.1100\ldots \cdot 2^{128-127=1} = -11.1 = -3.5_{\mathsf{TEN}}$

float to number (1)

1 sign bit

8 exponent bits ($2^{8-1}-1$ bias)

23 mantissa bits

`1 1000 0000 1100 0000 0000 0000 0000 000` $= ???$

$-1.1100\ldots \cdot 2^{128-127=1} = -11.1 = -3.5_{\text{TEN}}$

$10000000_{\text{TWO}} = 128_{\text{TEN}}$

float to number (1)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`1 1000 0000 1100 0000 0000 0000 0000 000` $= ???$

$-1.1100\ldots \cdot 2^{128-127=1} = -11.1 = -3.5_{\text{TEN}}$

$10000000_{\text{TWO}} = 128_{\text{TEN}}$

or $-1.11 \cdot 2^1 = -(2^0 + 2^{-1} + 2^{-2})2^1 = -(1.75) \cdot 2 = -3.5$

float to number (2)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0011 1001 0000 0000 0000 0000 000` $=$???

float to number (2)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0011 1001 0000 0000 0000 0000 000` $=$???

$+1.10010000\ldots \cdot 2^{131-127=4} = +1.1001 \cdot 2^4 = +11001 = 25_{\text{TEN}}$

float to number (2)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0011 1001 0000 0000 0000 0000 000` $= ???$

$+1.10010000\ldots \cdot 2^{131-127=4} = +1.1001 \cdot 2^4 = +11001 = 25_{\text{TEN}}$

float to number (2)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0011 1001 0000 0000 0000 0000 000` $=$???

$+1.10010000\ldots \cdot 2^{131-127=4} = +1.1001 \cdot 2^4 = +11001 = 25_{\text{TEN}}$

$10000011_{\text{TWO}} = 131_{\text{TEN}}$

float to number (2)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0011 1001 0000 0000 0000 0000 000` $=$???

$+1.10010000\ldots \cdot 2^{131-127=4} = +1.1001 \cdot 2^4 = +11001 = 25_{\text{TEN}}$

float to number (2)

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0011 1001 0000 0000 0000 0000 000` $= ???$

$+1.10010000\ldots \cdot 2^{131-127=4} = +1.1001 \cdot 2^4 = +11001 = 25_{\text{TEN}}$

or $(2^0 + 2^{-1} + 2^{-4})2^4 = (1 + .5 + .0625)16 = (1.5625)16 = 25$

`float` addition

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0000 1000 1000 0000 0000 0000 000` $+$

`0 0111 1111 0001 0000 0000 0000 0000 000` $=$???

float addition

1 sign bit

8 exponent bits ($2^{8-1}-1$ bias)

23 mantissa bits

`0 1000 0000 1000 1000 0000 0000 0000 000` $+$
`0 0111 1111 0001 0000 0000 0000 0000 000` $= ???$

$1.10001_{\text{TWO}} \cdot 2^1 + 1.0001 \cdot 2^0 = (11.0001 + 1.0001) \cdot 2^0 = 100.0010 \cdot 2^0 = 4.125_{\text{TEN}}$

`float` addition

1 sign bit

8 exponent bits ($2^{8-1} - 1$ bias)

23 mantissa bits

`0 1000 0000 1000 1000 0000 0000 0000 000` $+$
`0 0111 1111 0001 0000 0000 0000 0000 000` $= ???$

$1.10001_{\text{TWO}} \cdot 2^1 + 1.0001 \cdot 2^0 = (11.0001 + 1.0001) \cdot 2^0 = 100.0010 \cdot 2^0 = 4.125_{\text{TEN}}$

use difference between exponents to 'shift' mantissa; then add

floating point is not uniform

in half-precision, next number after:

$1 = 1.000\,000\,000\,0_{\text{TWO}} \cdot 2^0$ is $1.000\,000\,000\,1_{\text{TWO}} \cdot 2^0 \approx 1.0010_{\text{TEN}}$

$\sim +.001$

$100 = 1.100\,100\,000\,0_{\text{TWO}} \cdot 2^6$ is $1.100\,100\,000\,1_{\text{TWO}} \cdot 2^6 \approx 100.06_{\text{TEN}}$

$\sim +.06$

possible numbers are unevenly spaced

same as with 'normal' scientific notation:

$1 = 1.00 \cdot 10^0 \rightarrow 1.01 \cdot 10^0 = 1.01$ versus $1.00 \cdot 10^2 \rightarrow 1.01 \cdot 10^2 = 101$

don't compare with ==/!=

```
double x = 0.3;
double y = 0.1;
double y3 = y * 3;
if (x != y3) {
    cout << "not␣equal" << endl;
}
cout.setprecision(30);
cout << x << endl;
cout << y3 << endl;
```

```
not equal
0.299999999999999988897769753748
0.300000000000000044408920985006
```

on comparing floats

```
#include <cmath>
using std::fabs;
// or #include <math.h> and use fabs
    // without a using statement
...
    // chose based on expected accuracy
const float EPSILON = 1e-6;
float x, y;
...
if (fabs(x - y) < EPSILON) {
    ...
}
```

floating point accuracy

`float` — about 7 decimal places

`double` — about 15 decimal places

rounding errors (1)

$$2^{100} + 1$$

$2^{100} + 1$ cannot be represented exactly

- would need 99 mantissa bits
- rounds to 2^{100}

(but 2^{100} and 1 can)

rounding errors (2)

$$(2^{100}+1)-2^{100}$$

$$2^{100}-2^{100}$$

$$0$$

$$(2^{100}-2^{100})+1$$

$$0+1$$

$$1$$

dealing with rounding errors

avoid: adding and subtracting values of very different magntiudes

- tend to have big errors
- tend to have errors in one direction (compound over a calculation)

...by reordering and rearranging calculations

the problem of 0

0 is a very imporant number

can't be represented with implicit "1."

solution: special cases

IEEE float special cases

exponent bits	mantissa bits	meaning
`00000000`	`000...000`	± 0
`00000000`	non-zero	*denormal* number
`11111111`	`000...000`	$\pm\infty$
`11111111`	non-zero	not a number (NaN)

$$(+1/1000000000) \div \text{huge positive number} = +0$$
$$(-1/1000000000) \div \text{huge positive number} = -0$$
$$(+1000000000) \times \text{huge positive number} = +\infty$$
$$(-1000000000) \times \text{huge positive number} = -\infty$$
$$1 \div 0 = +\infty$$
$$0 \div 0 = \text{NaN}$$
$$\sqrt{-1} = \text{NaN}$$

float min magnitude value

exponent of `0000 0001` (not 0 since that's special)

mantisssa of `000...000`

$1.000000\ldots_{\textsf{TWO}} \cdot 2^{1-\textsf{bias}} = 2^{-126}$

float max magnitude value

exponent of `1111 1110` (not all 1s since that's special)

mantisssa of `111...111`

$1.111111\ldots11_{\text{TWO}} \cdot 2^{254-\text{bias}} = 1.11111\ldots1_{\text{TWO}} \cdot 2^{127} = 2^{128} - 2^{104}$

on denormals

denormals — minimum exponent bits, non-zero mantissa

smaller in magntiude than "normal" minimum value

- ignore the "implicit 1." rule

notorious for being superslow on some systems

- some CPUs take 100s of times longer to compute on them

we won't ask you about them

decimal floating point

if storing 0.001 exactly is important?

floating point formats base of 10 instead of 2

1.000×10^{-3}

example: IEEE decimal floating point

- 32, 64, 128-bit formats
- still store exponent$+$mantissa
- no leading "1." trick (doesn't work with 10^x)

binary-coded decimal

if integer conversion to/from base-10 is important?

but want to use binary hardware

one option: every 4 bits is a decimal digit

- not all possible bit patterns used

e.g. represent 147_{TEN} as `0001 0100 0111`

part of family on decimal-in-binary encodings

- some more compact than this (e.g. store 2 digits at a time)

backup slides

optimizing with overflow example

```
void foo(int x) {
    while (--x < 0) {
        bar();
    }
}
```

in latest version of `clang++` or `g++`

comples into an infinite loop if x is initially negative

if maximum optimizations are enabled (-O3 command-line option, not default)

IEEE single precision floating point

$$-1.1111\,1010\,0000\,0000\,0000\,000_{\text{TWO}} \cdot 2^{3}$$

IEEE single precision floating point

$$-1.1111\,1010\,0000\,0000\,0000\,000_{\text{TWO}} \cdot 2^{3}$$

sign (1 bit)

0 for $+$
1 for $-$

mantissa (23 bits)

don't store leading "1."
(because always present)

exponent (8 bits)

store $2 + 127 = 129$
127 is 'bias'

IEEE single precision floating point

$$-1.1111\,1010\,0000\,0000\,0000\,000_{\text{TWO}} \cdot 2^{3}$$

sign (1 bit)

mantissa (23 bits)

exponent (8 bits)

0 for $+$
1 for $-$

don't store leading "1."
(because always present)

store $2 + 127 = 129$
127 is 'bias'

```
1 100 0000 1 111 1101 0000 0000 0000 0000
```

IEEE single precision float

1 sign bit (1 for negative)

10 expontent bits

bias of 127 — if bits as unsigned are e, exponent is $E = e - 127$

23 mantissa bits

leading "1." not stored

value $= (1 - 2 \cdot \text{sign}) \cdot (1.\text{mantissa}_{\text{TWO}}) \cdot 2^{\text{exponent}-127}$