Big-Oh

asymptotic growth rate or *order*

compare two functions, but...

ignore constant factors, small inputs

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g grows faster — eventually much bigger than f

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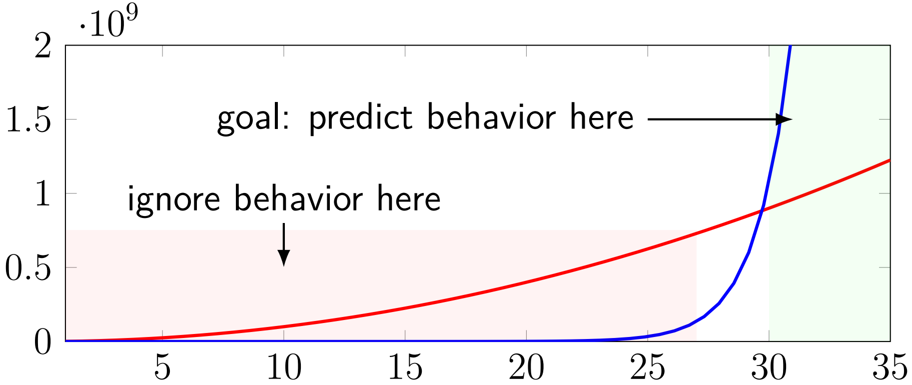

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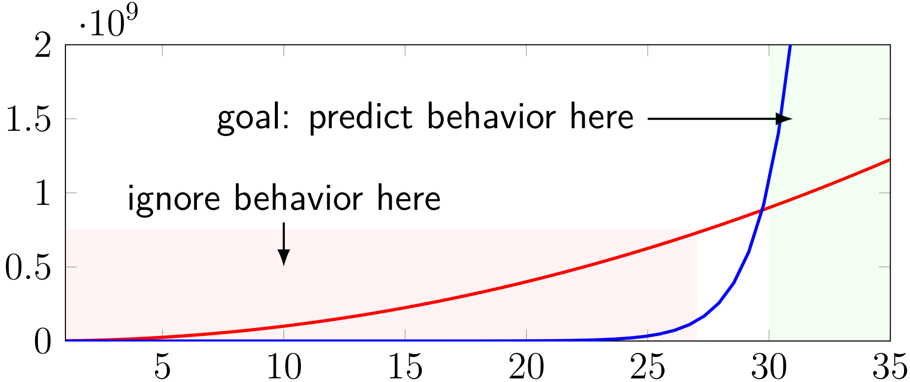

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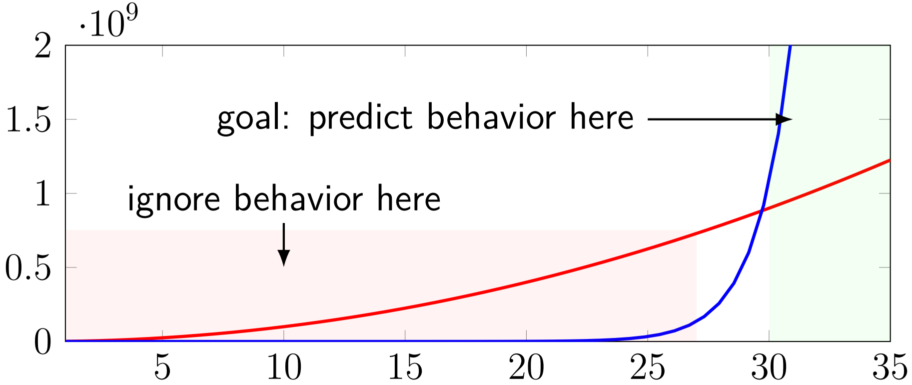

preview: what functions?

example: comparing sorting algorithms

runtime $= f$(size of input)

e.g. seconds to sort $= f$(number of elements in list)

e.g. # operations to sort $= f$(number of elements in list)

space $= f$(size of input)

e.g. number of bytes of memory $= f$(number of elements in list)

theory, not empirical

yes, you can make *guesses* about big-oh behavior from measurements

but, no, graphs $\neq$ big-oh comparison

- what happens further to the right?
- might not have tested big enough

want to write down formula

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- what happens further to the right?
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want to write down formula

example: summing a list of n items:

- exactly n addition operations
- *assume* each one takes k unit of time
- runtime $= f(n) = kn$

recall: comparing list data structures

List benchmark (from intro slides) w/ 100000 elements

Data structure	Total	Insert	Search	Delete
`Vector`	87.818	0.004	63.202	24.612 s
`ArrayList`	87.192	0.010	62.470	24.712 s
`LinkedList`	263.776	0.006	196.550	67.439 s
`HashSet`	0.029	0.022	0.003	0.004 s
`TreeSet`	0.134	0.110	0.017	0.007 s
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some runtimes get really big as size gets large...

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others seem to remain manageable

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problem: growth rate of runtimes with list size

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for `Vector` (unsorted), `ArrayList`, `LinkedList`...
$\#$ operations grows like n where n is list size

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for `HashSet`...
operations per search/remove is constant (sort of)

recall: comparing list data structures

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for `TreeSet`, sorted `Vector`...
$\#$ operations per search grows like $\log(n)$ where n is list size

why asymptotic analysis?

"can my program work when data gets big?"

- website gets thousands of new users?
- text editor opening 1MB book? 1 GB log file?
- music player sees $1\,000$ song collection? $50\,000$?
- text search on 100 petabyte copy of the text of the web?

why asymptotic analysis?

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text search on 100 petabyte copy of the text of the web?

if asymptotic analysis says "no"

can find out before implementing algorithm

won't be fixed by, e.g., buying a faster CPU

sets of functions

define sets of functions based on an example f

$\Omega(f)$: grow no slower than f ("$\geq f$")

$O(f)$: grow no faster than f ("$\leq f$")

$\Theta(f) = \Omega(f) \cap O(f)$: grow as fast as f ("$= f$")

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examples:

$n^3 \in \Omega(n^2)$
$100n \in O(n^2)$
$10n^2 + n \in \Theta(n^2)$ — ignore constant factor, etc.
and $10n^2 + n \in O(n^2)$ and $10n^2 + n \in \Omega(n^2)$

what are we measuring

$$
\begin{aligned}
f(n) &= \text{worst case running time} \\
n &= \text{input size — as a positive integer}
\end{aligned}
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will comapre f to another function $g(n)$

example: $f(n) \in O(g(n))$ (or $f \in O(g)$)
informally: "f is big-oh of g"

example $f(n) \notin \Omega(g(n))$ or ($g \notin \Omega(g)$)
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this class: almost always worst cases

intuition: detect if program will *ever* take "forever"

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example: iterating through an array until we find a value

- best case: look at one value, it's the one we want
- worst case: look at every value, none of them are what we want

worst case?

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example: iterating through an array until we find a value

best case: look at one value, it’s the one we want

worst case: look at every value, none of them are what we want

$f(n)$ is run time of *slowest* input of size n

formal definitions

$f(n) \in O(g(n))$:

there exists $c > 0$ and $n_0 > 0$ such that

for all $n > n_0$, $f(n) \leq c \cdot g(n)$

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$f(n) \in \Theta(g(n))$:
$f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$

formal definition example (1)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
$f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $n \in O(n^2)$:

formal definition example (1)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
$f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $n \in O(n^2)$:
choose $c = 1$, $n_0 = 2$
for $n > 2 = n_0$: $n \leq c \cdot n^2 = n^2$
Yes!

formal definition example (2)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
$f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $10n \in O(n)$?

formal definition example (2)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
$f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $10n \in O(n)$?

choose $c = 11$, $n_0 = 2$
for $n > 2 = n_0$: $f(n) = 10n \leq c \cdot g(n) = 11n$
Yes!

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$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
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Is $10n \in O(n)$?

choose $c = 11$, $n_0 = 2$
for $n > 2 = n_0$: $f(n) = 10n \leq c \cdot g(n) = 11n$
Yes!

don't need to choose smallest possible c

negating formal definitions

$f \in O(g)$: there exists $c, n_0 > 0$ so for all $n > n_0$: $f(n) \leq cg(n)$

$f \notin O(g)$:

there does not exist $c, n_0 > 0$ so for all $n > n_0$: $f(n) \leq cg(n)$
for all c, n_0, there exists $n > n_0$: $f(n) > cg(n)$

formal definition example (3)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
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Is $n^2 \in O(n)$?

formal definition example (3)

$f(n) \in O(g(n))$ if and only if

there exists $c > 0$ and $n_0 > 0$ such that

$f(n) \leq c \cdot g(n)$ for all $n > n_0$

Is $n^2 \in O(n)$?

no — consider any $c, n_0 > 0$

consider $n_{bad} = (c + 100)(n_0 + 100) > n_0$

$n_{bad}^2 = (c + 100)^2(n_0 + 100)^2 > c(c + 100)(n_0 + 100) = cn_{bad}$

so can't find c, n_0 that sastisfy definition

(i.e. $f(n) = n_{bad}^2 \not\leq c \cdot g(n_{bad}) = cn_{bad}$)

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so can't find c, n_0 that sastisfy definition
(i.e. $f(n) = n_{bad}^2 \not\leq c \cdot g(n_{bad}) = cn_{bad}$)

alternative

$n_{bad} = \max\{c + 100, n_0 + 1\} > n_0$

formal definition example (4)

$f(n) \in O(g(n))$ if and only if
there exists $c > 0$ and $n_0 > 0$ such that
$f(n) \leq c \cdot g(n)$ for all $n > n_0$

consider: $f(n) = 100 \cdot n^2 + n$, $g(n) = n^2$:
choose $c = 200$, $n_0 = 2$
observe for $n > 2$: $100n^2 + n \leq 101n^2$
for $n > 2 = n_0$: $f(n) = 100n^2 + n \leq 101n^2 \leq c \cdot g(n) = 200n^2$

big-oh proofs generally

if proving yes case:

- look at inequality
- *choose* a large enough c and n_0 that it's definitely true
- don't bother finding smallest c, n_0 that work

if proving no case:

- game: given c, n_0 find counter example
- general idea: *choose* $n > n_0$ using a formula based on c
- show that this n never satisfies the inequality
- don't bother showing it's true for all $n' > n$
- don't bother finding smallest n that works

aside: forall/exists

$\forall n > 0$: for all $n > 0$

$\exists n < 0$: there exists an $n < 0$

definition consequences

If $f \in O(h)$ and $g \notin O(h)$, which are true?

1. $\forall m > 0,\ f(m) < g(m)$

   for all m, f is less than g

2. $\exists m > 0,\ f(m) < g(m)$

   there exists an m, so f is less than g

3. $\exists m_0 > 0, \forall m > m_0,\ f(m) < g(m)$

   there exists an m_0, so for all m larger, f is less than g

4. 1 and 2

5. 2 and 3

6. 1 and 2 and 3

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If $f \in O(h)$ and $g \notin O(h)$, which are true?

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4. 1 and 2
5. 2 and 3
6. 1 and 2 and 3

$f \in O(h), g \notin O(h) \;\not\Rightarrow\; \forall m. f(m) < g(m)$

counterexample — $f(n) = 5n$; $g(n) = n^3$; $h(n) = n^2$

$f \in O(h)$: $5n \leq cn^2$ for all $n > n_0$ with $c = 6$, $n_0 = 2$
$g \notin O(h)$: $n^3 \leq cn^2$? use $n \approx cn_0$ as counterexample

$m = 2$: $f(m) = 10 \not< g(m) = 8$

$f \in O(h), g \notin O(h) \not\Longrightarrow \forall m. f(m) < g(m)$

counterexample — $f(n) = 5n$; $g(n) = n^3$; $h(n) = n^2$

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$m = 2$: $f(m) = 10 \not< g(m) = 8$

intuition: big-oh ignores behavior for small n

$n^3 \notin O(n^2)$

big-Oh definition requires:

$$n^3 \le cn^2 \text{ for all } n > n_0$$

choose any $c > 1$ and $n_0 > 1$, then

$$n = cn_0 \text{ is a counterexample}$$

$$n^3 = c^3 n_0^3 = cn_0(cn_0)^2 > cn^2$$

contradicting the definition
(and for $c < 1$, use $n = n_0 + 1$, etc.)

$$f \in O(h), g \notin O(h) \implies \exists m. f(m) < g(m)$$

intuition: should be true for 'big enough' m

assume definition of big-Oh:

$f \in O(h)$: $\forall n > n_0 : \; f(n) \leq ch(n)$ (for a $n_0, c > 0$)
$g \notin O(h)$: $\exists n > n_0 : \; g(n) > ch(n)$ (for *any* $n_0, c > 0$)

assume f's n_0, c

use the n that must exist for g (from definition)

$f \in O(h), g \notin O(h) \implies ?\exists m_0 \forall m > m_0 . f(m) < g(m)$

intuitively, seems so g must grow faster than f — for big m:

$$f(m) < c_1 \cdot h(m)$$
$$g(m) < c_2 \cdot h(m)$$

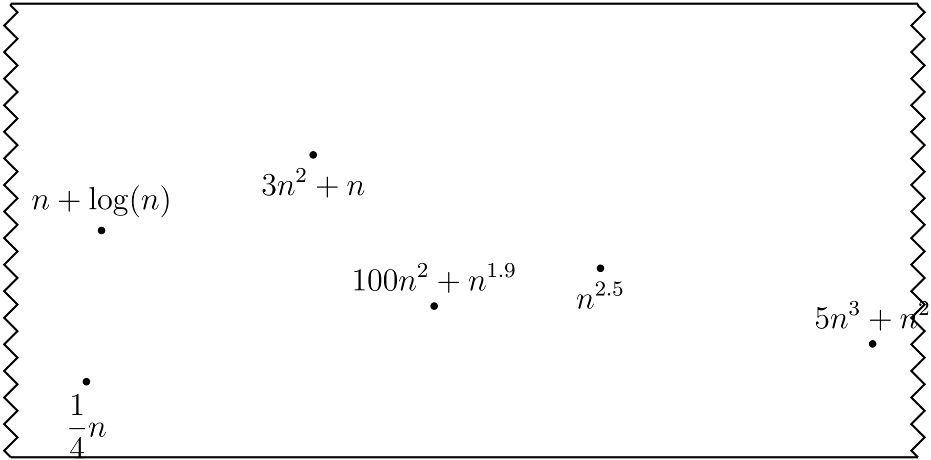
but some corner case counterexamples:

$$f(n) = n$$
$$g(n) = \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases}$$
$$h(n) = n$$

true with additional restriction:

f, g monotonic ($g(n) \leq g(n+1)$, etc.)

function hierarchy



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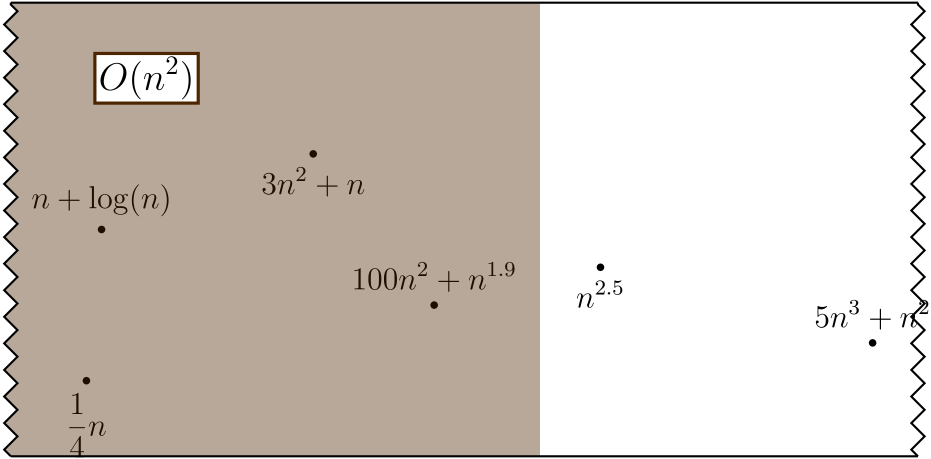
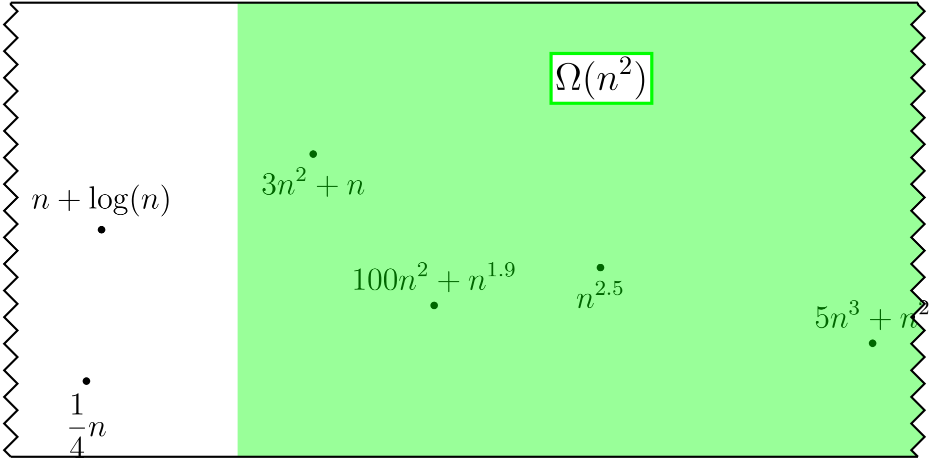


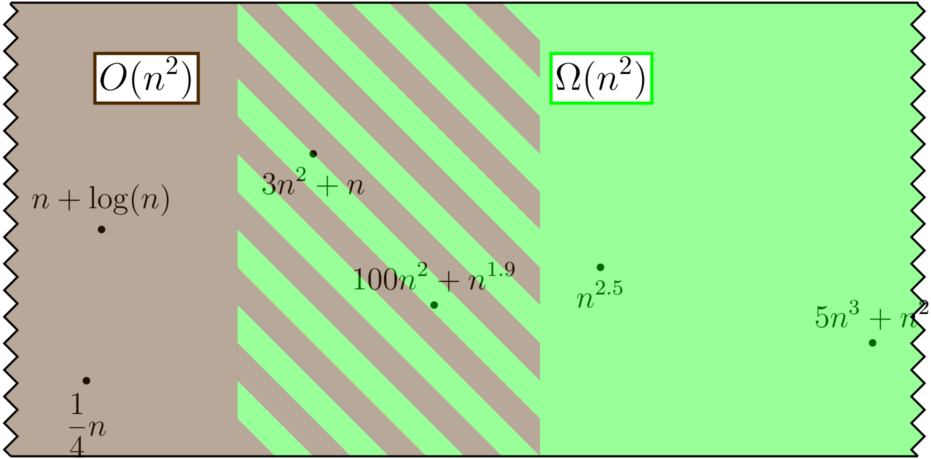
O — upper bound ("$\leq$")

function hierarchy



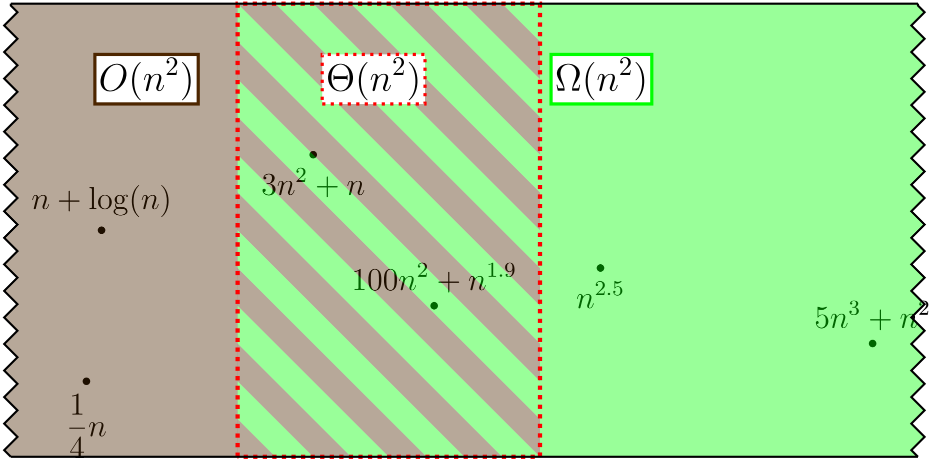
Ω — lower bound ("$\geq$")

function hierarchy



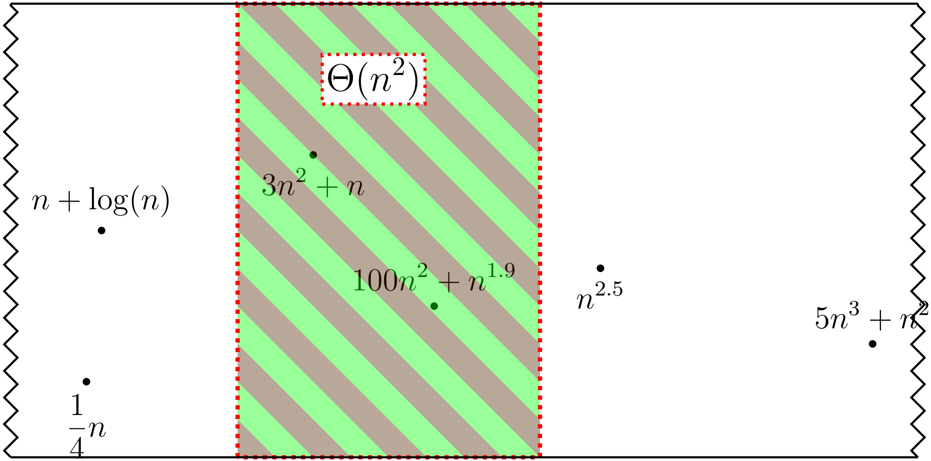
O and Ω overlap

function hierarchy



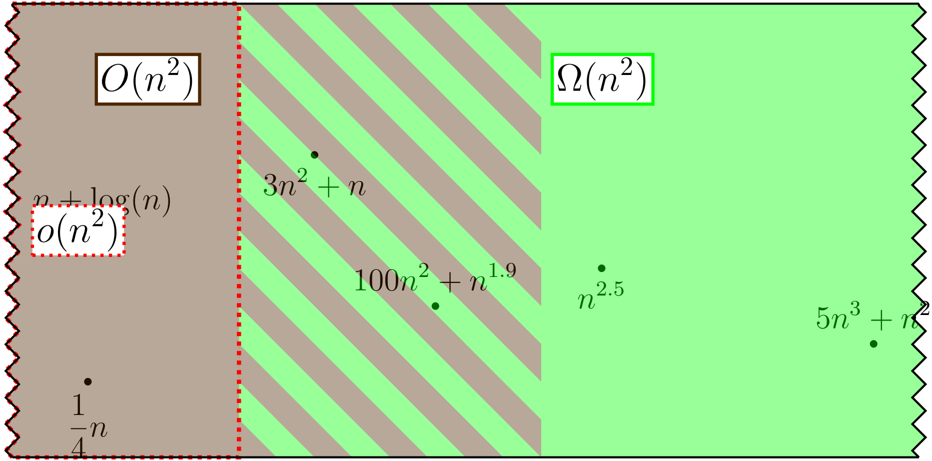
Θ — tight bound ("$=$") — O and Ω

function hierarchy



Θ — tight bound ("=") — O and Ω

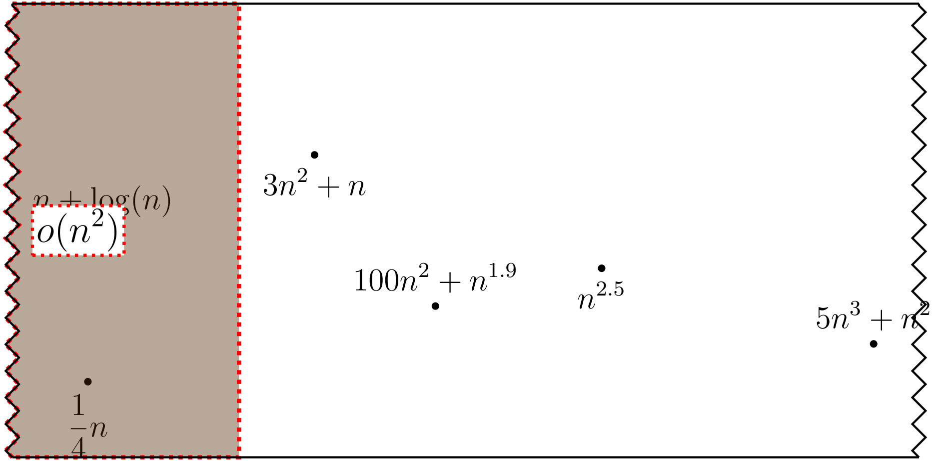
function hierarchy



$g \in o(f)$ ("little-oh")— strict upper bound

$f(n) < c \cdot g(n)$ (all c); (versus $O(f)$: $f(n) \leq c \cdot g(n)$)

function hierarchy



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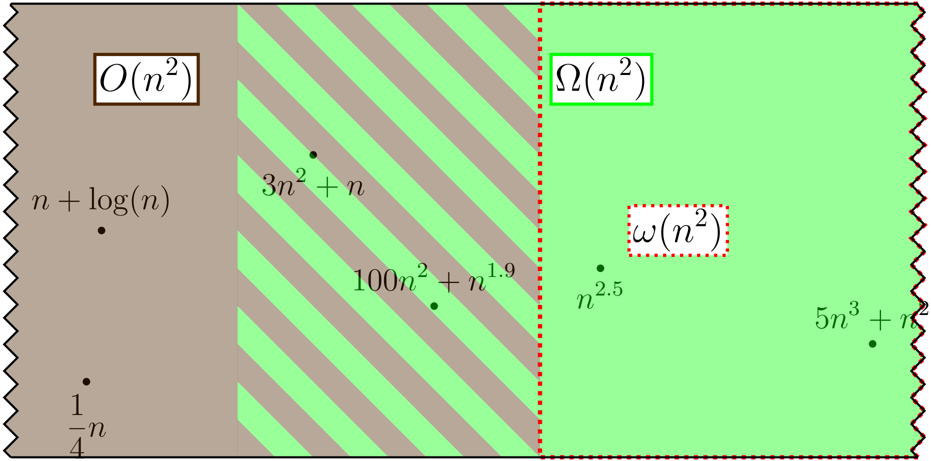

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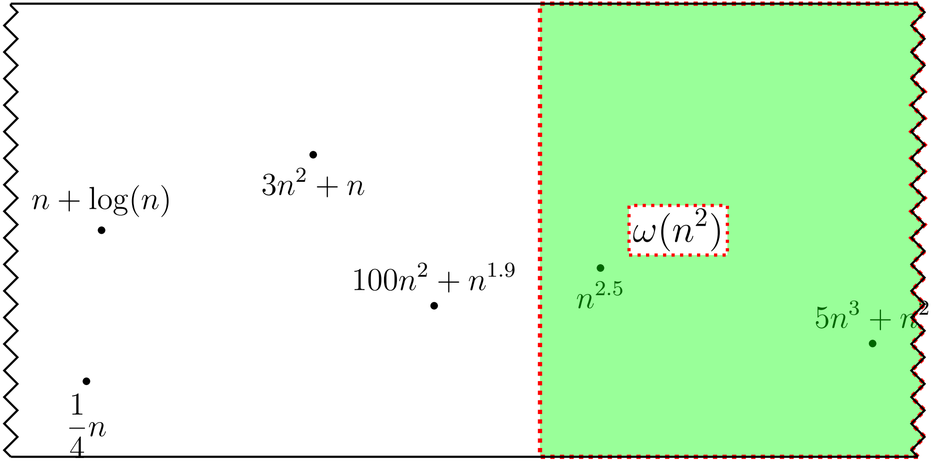

$g \in \omega(f)$ — strict lower bound

$f(n) > c \cdot g(n)$ (all c); (versus $\Omega(f)$: $f(n) \geq c \cdot g(n)$)

big-Oh variants

$O(f)$	asymptotically less than or equal to f
$o(f)$	asymptotically less than f
$\Omega(f)$	asymptotically greater than or equal to f
$\omega(f)$	asymptotically greater than f
$\Theta(f)$	asymptotically equal to f

limit-based definition

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} = X$$

if only if...

$X < \infty$: $f \in O(g)$

$X > 0$: $f \in \Omega(g)$

$0 < X < \infty$: $f \in \Theta(g)$

$X = 0$: $f \in o(g)$

$X = \infty$ (and $\liminf$): $f \in \omega(g)$

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lim sup?

$\limsup \frac{f(n)}{g(n)}$ — "limit superior"

equal to normal $\lim$ if it is defined

only care about upper bound

e.g. n^2 in $f(n) = \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases}$

usually glossed over (including in Bloomfield's/Floryan's slides from prior semesters)

some big-Oh properties (1)

for O and Ω and Θ:

$O(f+g) = O(\max(f,g))$

$f \in O(g)$ and $g \in O(h) \implies f \in O(h)$

also holds for o (little-oh), ω

$f \in O(f)$

some big-Oh properties (2)

$f \in O(g) \leftrightarrow g \in \Omega(f)$

$f \in \Theta(g) \leftrightarrow g \in \Theta(f)$

does *not* hold for O, Ω, etc.

Θ is an equivalence relation

reflexive, transitive, etc.

a note on =

informally, sometimes people write $5n^2 = O(n^2)$

not very precise — O is a *set of functions*

selected asymptotic relationships

for $k > 0$, $l > 0$, $c > 1$, $\epsilon > 0$:

$n^k \in o(c^{n^l})$ (polynomial always smaller than exponential)

$n^k \in o(n^k \log n)$ (adding $\log$ makes something bigger)

$\log_k(n) \in \Theta(\log_l(n))$ (all log bases are the same)

$n^k + cn^{k-1} \in \Theta(n^k)$ (only polynomial degree matters)

a note on logs

$$\log_k(n) = \frac{\log_l(n)}{\log_l(k)} = c \cdot \log_l(n)$$

therefore

$$\Theta(\log_k(n)) = \Theta(\log_l(n))$$

...so doesn't matter which base of log we mean

some names

$\Theta(1)$ — constant (some fixed maximum)

read kth element of array

$\Theta(\log n)$ — logarithmic

binary search a sorted array

$\Theta(n)$ — linear

searching an unsorted array

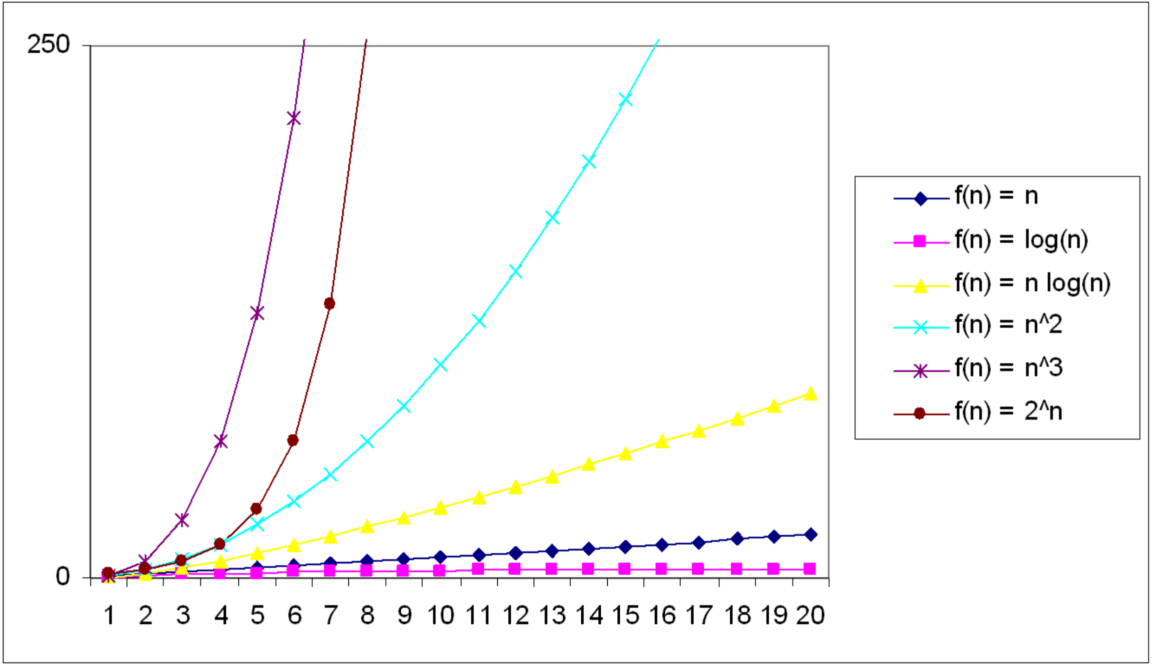
$\Theta(n \log n)$ — log-linear

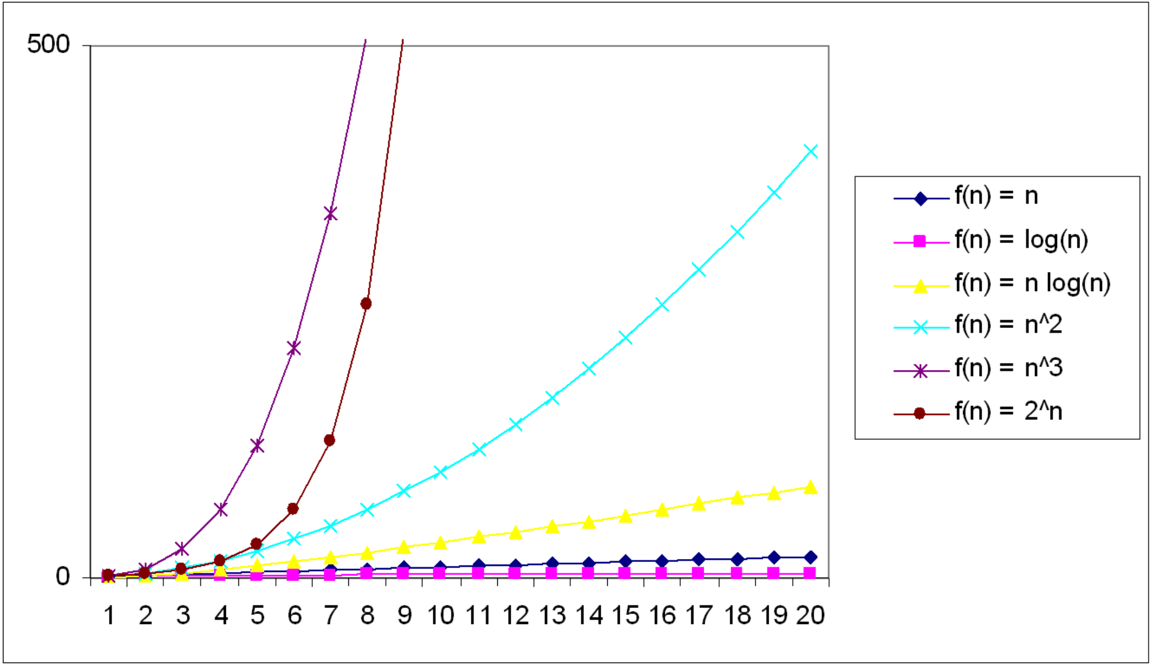
sorting an array by comparing elements

$\Theta(n^2)$ — quadratic

$\Theta(n^3)$ — cubic

$\Theta(2^n)$, $\Theta(c^n)$ — exponential


250
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
f(n) = n
f(n) = log(n)
f(n) = n log(n)
f(n) = n^2
f(n) = n^3
f(n) = 2^n


500
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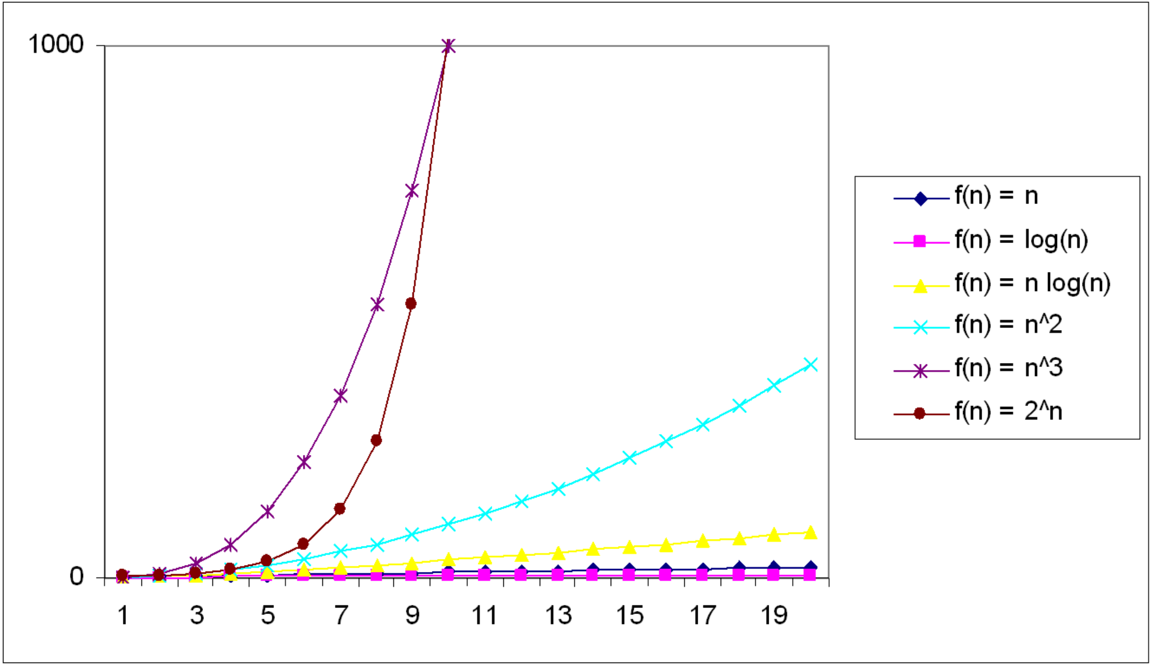
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
f(n) = n
f(n) = log(n)
f(n) = n log(n)
f(n) = n^2
f(n) = n^3
f(n) = 2^n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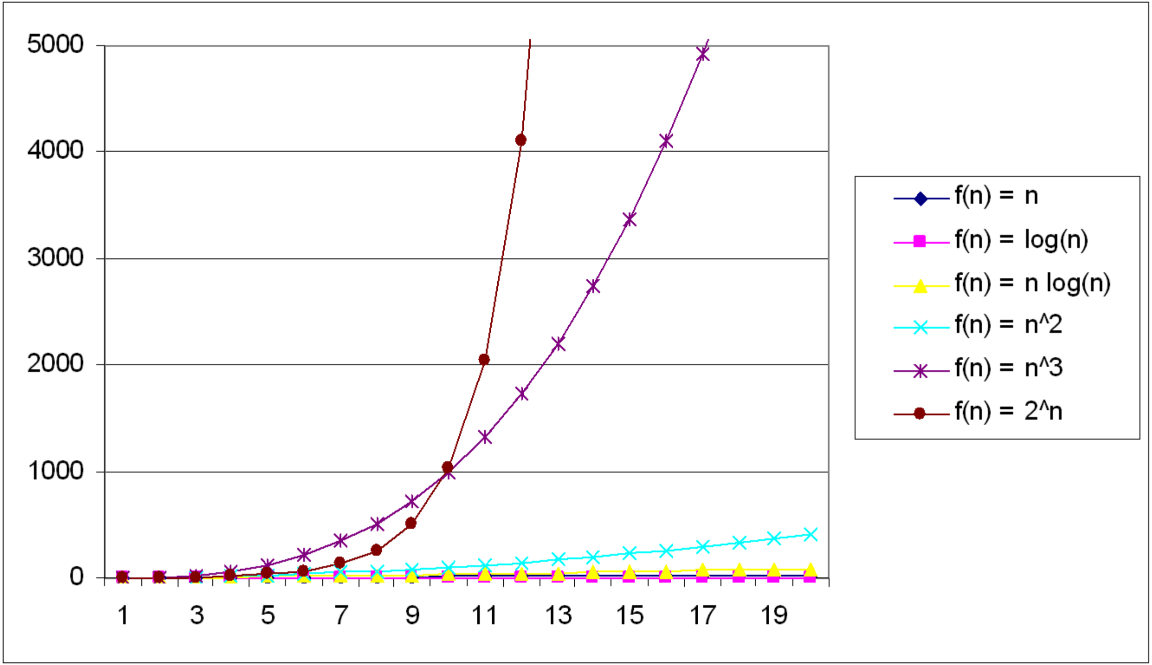
f(n) = n
f(n) = log(n)
f(n) = n log(n)
f(n) = n^2
f(n) = n^3
f(n) = 2^n



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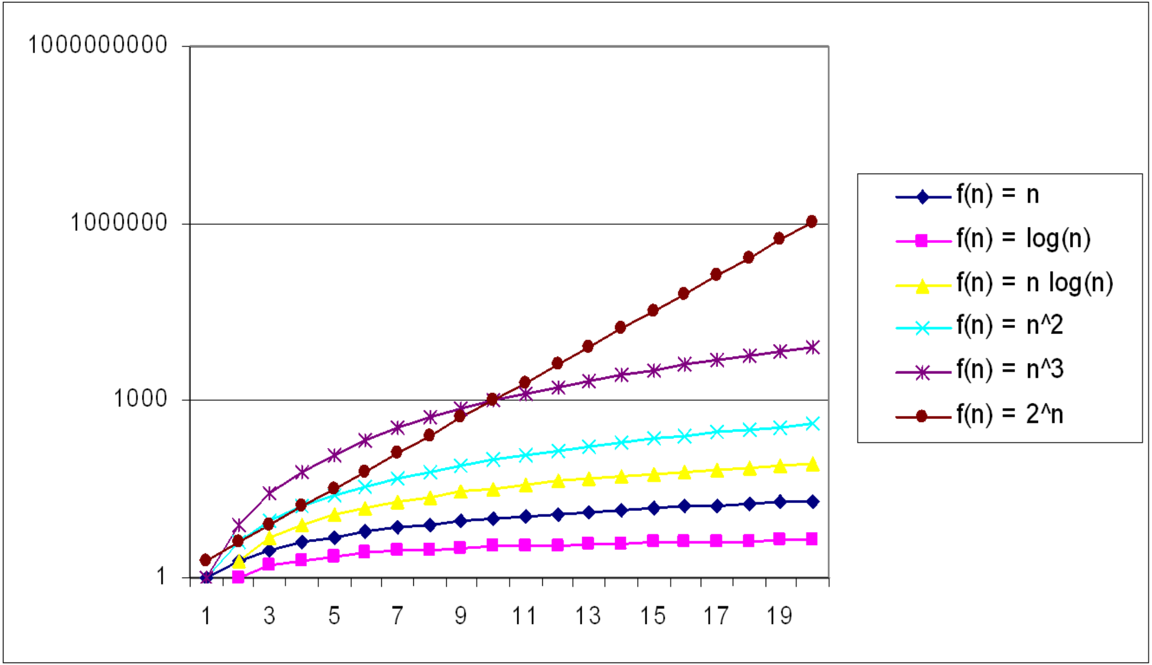
f(n) = n
f(n) = log(n)
f(n) = n log(n)
f(n) = n^2
f(n) = n^3
f(n) = 2^n



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19
f(n) = n
f(n) = log(n)
f(n) = n log(n)
f(n) = n^2
f(n) = n^3
f(n) = 2^n

big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of } \texttt{foo}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < M; ++j)
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of } \texttt{bar}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < i; ++j)
        foo();
```

runtime $\in \Theta\left(\sum_{i=0}^{N} i \times \text{runtime of } \texttt{foo}\right) = \Theta(N^2 \cdot \text{runtime of } \texttt{foo})$

big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of } \texttt{foo}))$

time to increment i?
"constant factor"
ignored by Θ

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < M; ++j)
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of } \texttt{bar}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < i; ++j)
        foo();
```

$$\text{runtime} \in \Theta\left(\sum_{i=0}^{N} i \times \text{runtime of } \texttt{foo}\right) = \Theta(N^2 \cdot \text{runtime of } \texttt{foo})$$

big-oh rules of thumb (1)

nested loops — work inside out
find time of inner loop (“foo”)
multiply by iterations of outer loop

```
for (int i = 0; i < N; ++i)
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of foo}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < M; ++j)
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of bar}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < i; ++j)
        foo();
```

runtime $\in \Theta\left(\sum_{i=0}^{N} i \times \text{runtime of foo}\right) = \Theta(N^2 \cdot \text{runtime of foo})$

big-oh rules of thumb (1)

```
for (int i = 0; i < N; ++i)
    foo();
```

runtime $\in \Theta(N \times (\text{runtime of foo}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < M; ++j)
        bar();
```

runtime $\in \Theta(N \times (M \times \text{runtime of bar}))$

```
for (int i = 0; i < N; ++i)
    for (int j = 0; j < i; ++j)
        foo();
```

at least $N/2$ iterations with
at least $N/2$ calls to foo
$\implies N/2 \cdot N/2 = N^2/4$
also $\leq N \cdot N = N^2$ calls
$\implies$ # calls to foo is $\Theta(N^2)$

runtime $\in \Theta\left(\sum_{i=0}^{N} i \times \text{runtime of foo}\right) = \Theta(N^2 \cdot \text{runtime of foo})$

big-oh rules of thumb (2)

```
foo();
bar();
```

runtime = runtime of foo + runtime of bar
$\in \Theta(\max\{\texttt{foo}\text{ runtime}, \texttt{bar}\text{ runtime}\})$

```
if (quux()) {
    foo();
} else {
    bar();
}
```

runtime $\approx$ runtime of quux + $\max$(runtime of foo, runtime of bar)
(max because we measure the worst-case)

$\Theta(1)$: constant time

constant time ($\Theta(1)$ time) — runtime does not depend on input

accessing an array element

linked list insert/delete (at known end)

getting a vector's size

...

is that really constant time

is getting vector's size really constant time?

vector stores its size, but, for, e.g. $N = 2^{10000}$, the size itself is huge

our *usual* assumption:

- treat "sensible" integer arithmetic as constant time
- (anything we'd keep in a `long` or smaller variable in practice?)

can do other analysis, but uncommon

- e.g. "bit complexity" — number of single bit operations

$\Theta(\log n)$: logarithmic time

binary search of sorted array

search space cut in half each iteration — $\lceil \log_2 N \rceil$ iterations

balanced tree search/insert

height of tree (somehow) gaurenteed to be $\Theta(\log N)$

$\Theta(n)$: linear

constant # operations/element

printing a list

search in unsorted array

search in linked list

doubling the size of a vector

unbalanced binary search tree find/insert

$\Theta(n \log n)$: log-linear

fast comparison-based sorting

merge sort, heap sort, ...

quicksort *if pivot choices are good*

inserting n elements into a balanced tree

$\Theta(n^2)$: quadratic

slow comparison-based sorting

insertion sort, bubble sort, selection sort, ...

quicksort *if pivot choices are bad*

most doubly nested for loops that go up to n

$\Theta(2^{n^c})$, $c \geq 1$: exponential

n-bit solution; try every 2^n of the possiblities

crack a combination lock by trying every possiblity

finding the best move in an $N \times N$ Go game (with Japanese rules)

checking satisfiablity of Boolean expression*

the Traveling Salesman problem*

*known algorithms — maybe can do better?

more?

$\Theta(n^3)$ — find shortest paths between all pairs of n nodes on a fully-connected graph

approx. order $2^{n^{1/3}}$ — best known integer factorization algorithm